## *Complexity of Algorithms*

### Example 1

Find a theta notation for the function $f(n) = 14n^3 + 6n^2\lg(n) + \lg(n) + 60$.
(Note that $n^3$ is the fastest growing term.)
Now

$$\begin{aligned} f(n) &\le 14n^3 + 6n^3 + n^3 + 60n^3 \\ &= 81n^3, \end{aligned}$$

ie $f(n) = O\left(n^3\right)$, with constant 81.
Also
$f(n) \ge 14n^3$, ie $f(n) = \Omega\left(n^3\right)$, with constant 14.

Since $f(n) = O\left(n^3\right)$, $f(n) = \Omega\left(n^3\right)$, then $f(n) = \Theta\left(n^3\right)$.
♦ ♦ ♦

### Example 2

Find a theta notation for the function $g(n) = 14\left(\lg(n)\right)^{60} + 5n + 2\lg(n) + 17$.
Note that the fastest term here is $n$, believe it or not. We said in class that lg($n$) grows more slowly than $n^\alpha$, for any positive $\alpha$. So $\lg(n)$ grows more slowly than $n^{\frac{1}{60}}$. Hence $\left(\lg(n)\right)^{60}$ grows more slowly than $\left(n^{\frac{1}{60}}\right)^{60} = n^{\frac{1}{60}*60} = n^1 = n$.

Hence

$$\begin{aligned} g(n) &\le 14n + 5n + 2n + 17n \\ &\le 38n, \end{aligned}$$

ie $g(n) = O(n)$, with constant 38.
Also
$g(n) \ge 5n$, ie $g(n) = \Omega(n)$, with constant 5.

Since $g(n) = O(n)$, $g(n) = \Omega(n)$, then $g(n) = \Theta(n)$.
♦ ♦ ♦

### Example 3

If $f(n) = \Theta\left(n^2\right)$ and $g(n) = \Theta\left(n^3\right)$, show that $f(n) + g(n) = \Theta\left(n^3\right)$.

Now from the facts given (ie the theta notations) we have the following.

$f(n) = O\left(n^2\right)$, ie $f(n) \le c_1 n^2,\ n \ge n_1$

$f(n) = \Omega\left(n^2\right)$, ie $f(n) \ge c_2 n^2,\ n \ge n_2$

$g(n) = O\left(n^3\right)$, ie $g(n) \le c_3 n^3,\ n \ge n_3$

$g(n) = \Omega\left(n^3\right)$, ie $g(n) \ge c_4 n^3,\ n \ge n_3$

> Note that the definitions of O, $\Theta$ notation I use here do not involve the modulus function, eg I have defined
> $f(n) = O\left(g(n)\right)$ if $f(n) \le c_1 g(n)$, etc.

for some suitable (positive) constants $c_1$, $c_2$, $c_3$, $c_4$, $n_1$, $n_2$, $n_3$ and $n_4$.

Now we have

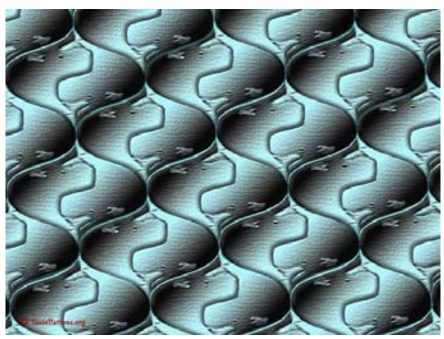
$f(n) + g(n) \le c_1 n^2 + c_3 n^3 \le c_1 n^3 + c_3 n^3 = (c_1 + c_3) n^3$, provided $n \ge n_1, n_3$.

So $f(n) + g(n) = O\left(n^3\right)$, with constant $c_1$+$c_3$. Also

$f(n) + g(n) \ge c_2 n^2 + c_4 n^3 \ge c_4 n^3 = (c_4) n^3$.

So $f(n) + g(n) = \Omega\left(n^3\right)$, with constant c4.

Hence $f(n) + g(n) = \Theta\left(n^3\right)$.



## Its Your Turn…

Now try these questions, using what you have learnt from the first examples..

1. Pick the fastest growing term in each of the following. State the theta notation.

(a) $f(n) = 4n^3 + 2n^2 + 50 * \lg(n) + 10$

(b) $f(n) = 12n^2 + 12000 * \lg(n) + 3.2^n$

(c) $f(n) = 200n^{10} + 12\lg(n) + 4n!$

2. Use the definitions of big ‘Oh’ and big ‘Omega’ to state suitable inequalities in each of the following. Declare any constants you are using.

(a) $f(n) = O\left(n^2\right)$

(b) $g(n) = \Omega\left(n \lg(n)\right)$

(c) $h(n) = O\left(2^n\right)$

3. Order these functions from fastest growing to slowest growing. Which would be the most desirable to have as theta notation for an algorithm? Which would be least desirable?

$n^2, \lg(n), 2^n, n^{1.5}, n!, 3^n$

4. Complete the following lines of argument finding the appropriate theta notation, and the relevant constants.

(a)

$f(n) = 14n\lg(n) + 2n^2 + 4n + 3$

$f(n) \leq \qquad\qquad\qquad = 23n^2$

ie $f(n) = \mathrm{O}\left(n^2\right)$ with $c_1 = 25$

(b)

$g(n) = 4n^3 + 3.2^n + 10\lg(n)$

$g(n) \geq \qquad\qquad\qquad = \quad .2^n$

ie $g(n) = \Omega\left(2^n\right)$ with $c_2 = \qquad .$

(c)

$h(n) = 3n^2\lg(n) + 40n^{20} + 10n\lg(n) + 400$

$h(n) \leq \qquad\qquad\qquad\qquad =$

ie $h(n) = \mathrm{O}( \qquad ),$ with $c_1 = \qquad .$

5. By using the formula $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$, find the fastest growing term for the left hand side. Once you have found the fastest growing term, state the theta notation without a full proof.

Now try and find the theta notation for the following (answers below).

1. $h(n) = 4n! + 3*2^n + 50n^{100}$
2. $i(n) = 12n\lg(n) + 6n^6 + 17n^2 + 11$
3. $j(n) = 6n\lg(n) + 90n + 200\lg(n) + 3$

1. $h(n) = \Theta\left(n!\right), c_1 = 57, c_2 = 4$
2. $i(n) = \Theta\left(n^6\right), c_1 = 46, c_2 = 6$
3. $j(n) = \Theta\left(n\lg(n)\right), c_1 = 299, c_2 = 6$