Tutorial 2 Solutions

Preparation Questions

1.4

2. $\forall x \exists y T(x, y)$ For all students *x* in the group, there is a student *y* in the group shorter than *x*. This is false. Let *x*=Erin, then there is no one in the group shorter than Erin. Hence no such *y* exists when *x*= Erin.

3. $\exists x \forall y T(x, y)$ There is a student *y* in the group, such that for all students *x* in the group, *y* is taller than *x*. False, as there is no student who is taller than everyone in the group. (Marty is taller than everyone else in the group, but he is not taller than himself.)

1.5

15. Place the 100 balls in the 9 boxes. Assume that no box has more than 11 balls in it. Then the maximum number of balls that I can have placed in the boxes is 11*9=99. But this is a contradiction, because I placed 100 balls to start with. Hence my assumption that no box has more than 11 balls must be incorrect. Then some box has 12 or more balls.

31. *p*: I study hard, *q*: I get A's, *r*: I get rich

H_1	*If* I study hard *then* I get A's	$p \to q$
H_2	I study hard	p
C	$\therefore$ I get A's	q

		H_1	H_2	**C**
p	*q*	$p \to q$	*p*	*q*
T	T	***T***	***T***	***T***
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

The first row is the only row in which all the hypotheses are true. The conclusion is also true, so we conclude this is a valid argument.

39
p: 4 Megabytes is better than no memory at all.
q: We will buy more memory
r: We will buy a new computer

H_1 $\neg r \to \neg p$ If we do not buy a new computer then 4 megabytes is not better than no memory at all

H_2	r	We will buy a new computer
C	$\therefore p$	$\therefore$ 4 megabytes is better than no memory at all.

					H_1	H_2	C
p	q	r	$\neg r$	$\neg p$	$\neg r \rightarrow \neg p$	r	p
T	T	T	F	F	***T***	***T***	***T***
T	T	F	T	F	F	F	T
T	F	T	F	F	***T***	***T***	***T***
T	F	F	T	F	F	F	T
F	T	T	F	T	***T***	***T***	***F***
F	T	F	T	T	T	F	F
F	F	T	F	T	T	T	F
F	F	F	T	T	T	F	F

Invalid argument, row 5 has both hypotheses true but conclusion false.

1.7

1. <u>BS (n=1)</u>
LHS=1
RHS=1^2=1=LHS.

<u>IS</u>
Assume $1+3+5+\cdots+(2n-1)=n^2$.

Try to prove $1+3+5+\cdots+(2n-1)+(2n+1)=\left(n+1\right)^2$.

Now

$$\begin{aligned} LHS &= 1+3+5+\cdots+(2n-1)+(2n+1) \\ &= n^2+(2n+1) \quad \longleftarrow \boxed{\text{By our inductive assumption.}} \\ &= (n+1)^2 \\ &= RHS \end{aligned}$$

Proved.

21. <u>BS (n=1)</u>
Now $7^1-1=6$, which is divisible by 6.
(Remember that the answer to the question "Is 7^n-6 divisible by 6?" is true or false, ie a Boolean, not a number. The definition of divisibility is that if *A* is divisible by *B* then there exists an integer *k* with *A*=*B***k*.)

<u>IS</u>
Assume that for some integer $n \geq 1$, 7^n-1 is divisible by 6, ie there exists an integer *k* with $7^n-1=6*k$, for some integer *k*.
Try to prove that 7^{n+1}-1 is also divisible by 6, ie that $7^{n+1}-1=6j$, for some integer *j*.
Now $7^{n+1}-1=7*7^n-1=(6+1)*7^n-1=6*7^n+7^n-1=6*7^n+6k=6(7^n+k)$, which is divisible by 6.
Proved.

Tutorial Questions

1.4

28. $\forall x \exists y\left(x^2 + y^2 = 9\right)$ False, eg *x*=5. Then $y^2 = -16$, which is impossible. Hence there is no such *y* for this *x* value.

1.5

34.
H1 *If* I study hard or I get rich, *then* I get A's $(p \vee r) \to q$
H2 I get A's q
C $\therefore$ *If* I don't study hard *then* I get rich $\neg p \to r$

				H_1	H_2			C
p	*q*	*r*	$p \vee r$	$(p \vee r) \to q$	*q*	$\neg p$	*r*	$\neg p \to r$
T	T	T	T	***T***	***T***	F	T	***T***
T	T	F	T	***T***	***T***	F	F	***T***
T	F	T	T	F	F	F	T	T
T	F	F	T	F	F	F	F	T
F	T	T	T	***T***	***T***	T	T	***T***
F	T	F	F	***T***	***T***	T	F	***F***
F	F	T	T	F	F	T	T	T
F	F	F	F	T	F	T	F	F

This is an invalid argument, as row 6 shows both hypotheses true, but the conclusion is false.

44.
H_1 $p \to (q \to r)$
H_2 $q \to (p \to r)$
C $\therefore (p \vee q) \to r$

				H_1		H_2		C
p	*q*	*r*	$q \to r$ *(A)*	$p \to A$	$p \to r$ *(B)*	$q \to B$	$p \vee q$ *(D)*	$D \to r$
T	T	T	T	***T***	T	***T***	T	***T***
T	T	F	F	F	F	F	T	F
T	F	T	T	T	T	T	T	T
T	F	F	T	***T***	F	***T***	T	***F***
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	F

F	F	T	T	T	T	T	F	T
F	F	F	T	T	T	T	F	T

Invalid argument, as row 4 has both hypotheses true but the conclusion false.

1.7

2. <u>BS (*n*=1)</u>

$$LHS = 1.2 = 2$$

$$RHS = \frac{1*2*3}{3} = 2 = LHS$$

<u>IS</u>

Assume $1.2 + 2.3 + 3.4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Try to prove $1.2 + 2.3 + 3.4 + \cdots + n(n+1) + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}$.

Now

$$LHS = 1.2 + 2.3 + 3.4 + \cdots + n(n+1) + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}$$

$$= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$$

$$= (n+1)(n+2)\left[\frac{n}{3} + 1\right]$$

$$= (n+1)(n+2)\frac{n+3}{3}$$

$$= \frac{(n+1)(n+2)(n+3)}{3}$$

$$= RHS$$

Proved

12. <u>BS (*n*=1)</u>

$$LHS = \frac{1}{2*1} = \frac{1}{2}$$

$$RHS = \frac{1}{2}$$

Hence *LHS*≤*RHS*.

<u>IS</u>

Assume $\frac{1}{2n} \le \frac{1*3*5*\cdots*(2n-1)}{2*4*6*\cdots*(2n)}$.

Try to prove $\frac{1}{2(n+1)} \le \frac{1*3*5*\cdots*(2n-1)*(2n+1)}{2*4*6*\cdots*(2n)*(2n+2)}$.

(Note that of course 2(*n*+1)=2*n*+2, on the LHS, and adding 2 to 2*n*-1 gives 2*n*+1 on the RHS, and adding 2 to 2*n* gives 2*n*+2 also.)

Now

$$RHS=\frac{1*3*5*\cdots*(2n-1)*(2n+1)}{2*4*6*\cdots*(2n)*(2n+2)}$$

$$=\frac{1*3*5*\cdots*(2n-1)}{2*4*6*\cdots*(2n)}*\frac{2n+1}{2n+2}$$

$$\geq\frac{1}{2n}*\frac{2n+1}{2n+2}$$

$$=\frac{1}{2n+2}*\frac{2n+1}{2n}$$

Noting that (2n+1)/(2n)≥1.

$$\geq\frac{1}{2n+2}=LHS$$

$$\text{ie } LHS\leq RHS$$

Proved.

14. BS (n=3)

$$LHS=2*3+1=7$$

$$RHS=2^3=8$$

$$LHS\leq RHS$$

IS

Assume $2n+1\leq 2^n$.

As always when we use the weak form, this is assumed only for *some* (ie only 1) unknown value of n, here ≥3.

Try to prove $2(n+1)+1=2n+2+1\leq 2^n$

Now

$$LHS=2n+2+1\leq 2^n+3$$

$$\leq 2^n+2^n=2*2^n=2^{n+1}=RHS$$

Proved.

(Note: I knew that I was looking for 2^{n+1}, which is $2*2^n$. OF course $A+A=2A$, so I also knew that $2*2^n$ could be written as 2^n+2^n, which turned out to be easy to find. This points up the need to review your powers and algebra.)

1.8

2. B.S. (n=24, 24, 26, 27, 28)

(Note that I need as many values in my basis step as the value of my smallest stamp. This is because to get to the value n+1c, I have to add a 5c or 7c stamp. Hence the value I am adding a 5c stamp to must be n-4, so that (n-4)+5=n+1. So I will need 5 values, 24, 25, 26, 27 and 28, with 28 being the smallest possible n value in my inductive step.)

24c=2*5c+2*7c
25c=5*5c
26c=1*5c+3*7c

27c=4*5c+1*7c
28c=4*7c

IS

Suppose that postage of 24, 25, 26, 27, 28,…,*n* cents can be made up using only 5c and 7c stamps. We seek to prove that postage of n+1c can be made up using only 5c and 7 c stamps.

Now *n*+1=(*n*-4)+5, so by adding a 5c stamp to the postage of *n*-1c, we can make up *n*+1c using only 5c and 7c stamps. Proved.