Solution to Tute 2, Section 6.2, Question 51

The barrel is obtained by rotating the curve $y = f(x) = R - kx^2, \frac{-h}{2} \le x \le \frac{h}{2}$ about the x axis. At the ends of the barrel, the radius of the barrel will be equal to the height of the function,

$$f\left(\frac{h}{2}\right) = R - k\left(\frac{h}{2}\right)^2 = R - \frac{kh^2}{4}.$$

Of course, if $\delta = \frac{kh^2}{4}$, then the end radius is $R - \delta$.

Now we find the volume for $0 \le x \le \frac{h}{2}$ and double it. As usual, if we rotate a thin vertical sliver around the x axis, we obtain (roughly) a thin disk. So an elemental volume is given by $dV_i = \pi r_i^2 dx = \pi(R - kx_i^2)dx$.

Note that $\delta = kh^2/4$.

So, forming a Riemann sum, we obtain $V \approx \sum_{i=1}^{n} dV_i = \sum_{i=1}^{n} \pi\left(R - kx_i^2\right)^2 dx$, and in the limit this becomes

$$\begin{aligned}
V &= \int_0^{\frac{h}{2}} \pi\left(R - kx^2\right)^2 dx \\
&= \int_0^{\frac{h}{2}} \pi\left(R^2 - 2kRx^2 + k^2x^4\right) dx \\
&= \pi\left[R^2x - \frac{2}{3}kRx^3 + \frac{1}{5}k^2x^5\right]_0^{\frac{h}{2}} \\
&= \pi\left[R^2\frac{h}{2} - \frac{2}{3}kR\frac{h^3}{8} + \frac{1}{5}k^2\frac{h^5}{32}\right] - \pi[0] \\
&= \pi h\left[\frac{R^2}{2} - \frac{kRh^2}{12} + \frac{k^2h^4}{160}\right]
\end{aligned}$$

Hence the full volume will be double this.

$$
\begin{aligned}
V_T &= \pi h\left[R^2 - \frac{kRh^2}{6} + \frac{k^2h^4}{80}\right] \\
&= \pi h\left[R^2 - \frac{2R}{3}\frac{kh^2}{4} + \frac{1}{80}k^2h^4\right] \\
&= \pi h\left[R^2 - \frac{2R}{3}\delta + \frac{1}{80}*16\delta^2\right] \\
&= \frac{\pi h}{3}\left[3R^2 - 2R\delta + \frac{3}{5}\delta^2\right] \\
&= \frac{\pi h}{3}\left[2R^2 + R^2 - 2R\delta + \frac{3}{5}\delta^2\right]
\end{aligned}
$$

Note that $-2R\delta$ is a term from $(R-\delta)^2 = r^2$.

And if $r = R - \delta, r^2 = R^2 - 2R\delta + \delta^2$.

So
$$
\begin{aligned}
V_T &= \frac{\pi h}{3}\left[2R^2 + R^2 - 2R\delta + \delta^2 - \delta^2 + \frac{3}{5}\delta\right] \\
&= \frac{\pi h}{3}\left[2R^2 + r^2 - \frac{2}{5}\delta^2\right]
\end{aligned}
$$

Voila!