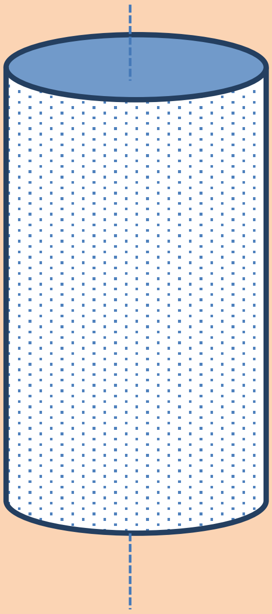
Constructing Moment of Inertia

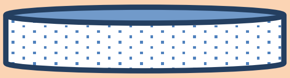
In building a Riemann sum to find the moment of inertia of a solid object it is essential that the body be divided into elemental volumes in which each part of the volume is at approximately the same distance from the axis of rotation. Of course the element will always be of some finite thickness, say *dx* or *dy* or *dz*, and so not all parts can be exactly the same distance from the axis of rotation. Nevertheless, no part of the element should be more than *dx* (or *dy* or *dz*) further from the axis than any other point. In all examples assume the density of the object is ρ.

Example 1

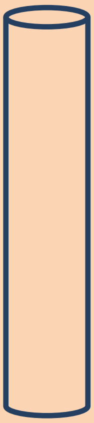
A cylinder of height *h* and radius *R* is rotating about its long axis. Find the elemental volumes, and the moment of inertia before setting up the Riemann sum.



We could try a horizontal disc of height *dh*, radius *R*.



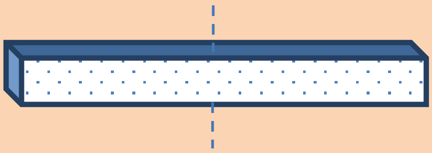
However near the central axis the distance to the axis of rotation is nearly 0, but at the edge of the cylinder it is *R*. Hence this cannot be a suitable element. Try a cylindrical shell of radius *r*, width *dr*, height *h*.



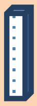
Because the shell is very thin, every point in the shell is at about the same distance from the central axis. The moment of inertia of this shell is then given by $mr^2 = \rho.2\pi r.h.dr.r^2 = 2\pi\rho r^3 hdr$.

Example 2

Suppose we wish to find the moment of inertia of a rod with rectangular cross section rotating about an axis perpendicular to its long axis, through its middle. (This is similar to the rod used as an example in class, but here the axis of rotation is through the middle of the rod.) The height of the rod is *h*, its length is *L* and its thickness is *w*.



In this case a vertical strip, at right angles to the long axis of the rod, will be at approximately the same distance of the axis of rotation. Suppose the strip is of width *dx*, and is located at distance *x* from the axis of rotation.



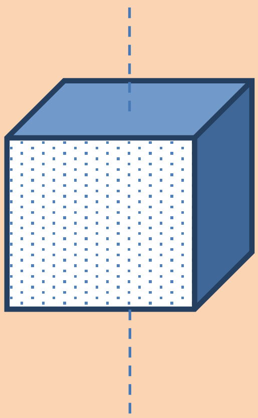
The mass of this strip is $\rho hwdx$ and so its moment of inertia is $\rho hwdx * x^2 = \rho hwx^2 dx$.

Add these strips up before integrating from $\frac{-L}{2}$ to $\frac{L}{2}$.

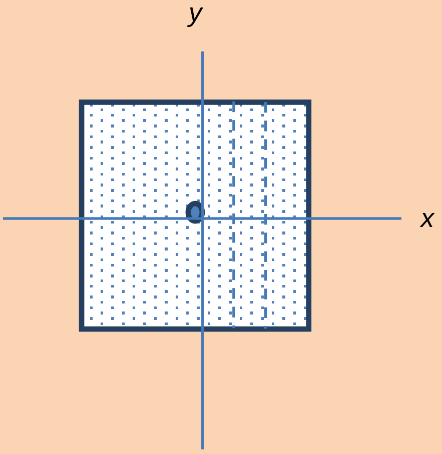
Example 3

Suppose we wish to find the moment of inertia of a cube of side *D* about an axis through the centre of two opposing faces. This is very tricky, because if we use cylindrical slices, then near the edges of the cubes the shells will contact the faces and pass into space. Hence a shell is not appropriate.

Let the axis of rotation be the *z* axis.



In this case we should form horizontal slices and find the moment of inertia of each of these. The diagram below is the top view of such a slice, with the axis of rotation through the middle. However this is itself not a simple problem. We need to use the parallel axes theorem, to find the moment of inertia of a thin strip. In other words, use the result from class, or an integration, to find the moment of inertia of the thin strip about its own centre and the use the parallel axes theorem to find the moment about the centre of the slice.

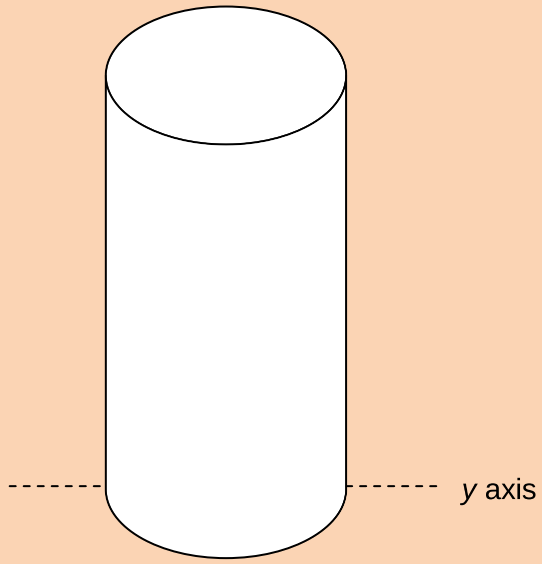


Suppose the strip is of width *dx* and is at distance *x* from the *y* axis. This is equivalent to two rods of length $\frac{1}{2}D$ rotating about their ends. From lectures we know that the moment of inertia of two such rods must be $\dfrac{2.dz.dx\rho*\left(\dfrac{D}{2}\right)^3}{3}=\dfrac{\rho D^3 dzdx}{12}$.

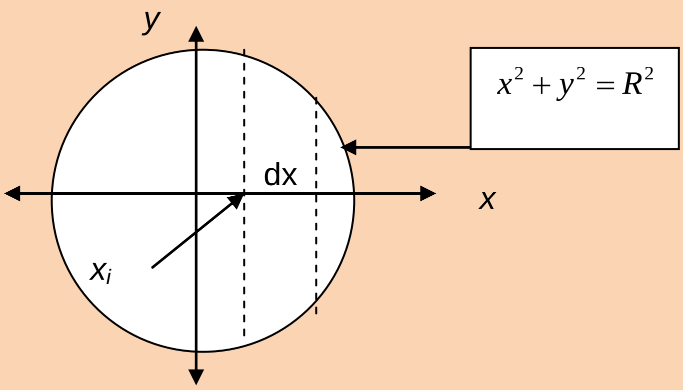
Now to find the moment of inertia about the centre of the horizontal slice, we add to this result $mass*x^2$. An integration over the slice will then give us the moment of inertia of the whole slice, and an integration in the *z* direction will give us the moment of inertia of the whole cube.

Example 4

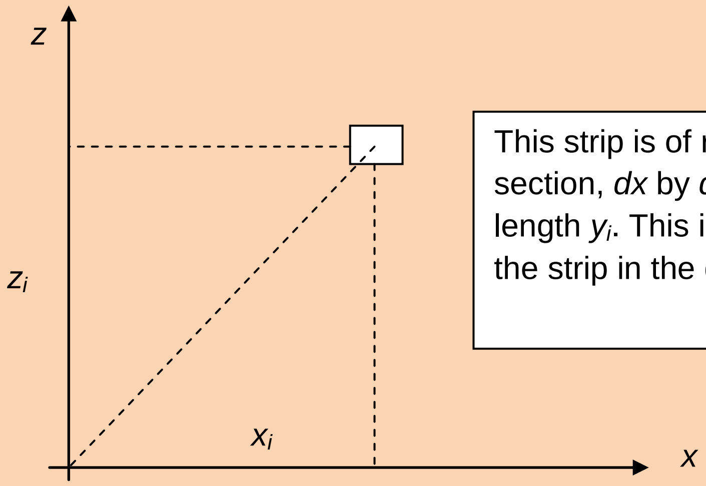
Now imagine we need to find the moment of inertia of a cylinder rotating about a diameter of its base. As before the radius is *R* and the height *h*.



This is complicated, since we can't use a vertical slice, a horizontal slice or a cylindrical shell to find the elemental moment of inertia, just like the last problem. Instead we take a horizontal slice of thickness *dz*, at a height of z_i, and try to find the moment of inertia of that via integration.



Now take a thin slice of width *dx* at position x_i on the *x* axis. This is nearly a rectangular strip, and because it is parallel to the axis of rotation, every point in it is approximately the same distance from the axis of rotation.



This strip is of rectangular cross section, *dx* by *dz*, and it is of length y_i. This is the end view of the strip in the diagram above.

So the distance to the axis of rotation is given by $\sqrt{x_i^2 + z_i^2}$ and hence the moment of inertia of the strip is given by the mass of the strip multiplied by the square of the distance to the axis of rotation, which is $x_i^2 + z_i^2$. Now this needs to be integrated in the x direction from $-R$ to $+R$, and in the z direction from 0 to h.