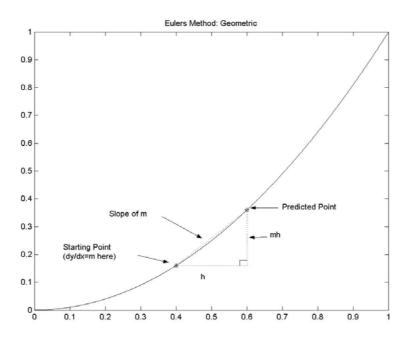| **Numerical Solution to Ordinary Differential Equations** |
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We frequently need to solve a differential equation, ie an equation of the form $\frac{dy}{dx} = f(x, y); a \le x \le b; y(a) = \alpha$. In other words, we are not given the function y but its derivative. This derivative may be only a function of x, but it could be a function of x and y. We are also given the value of y at one end of the domain, ie it is what is called an *initial value problem*. There are many very sophisticated ways to answer this problem, but we shall examine a fairly simple one, and come at it from several different ways[1]. The final result will be a table of values for y at various x values. This will enable us to, for example, plot the graph of y as a function of x, or simply to estimate the y value at some x value we need. In this case we will know the y value at many x values besides the one we are interested in, but there is no way around this.

Approach 1: Geometric.

Consider the diagram below. We know that the derivative, $\frac{dy}{dx}$, is given by the function $f(x,y)$. Hence if we wish to know the value of the derivative or slope at $x=a$, then we substitute the known values of a and α into f. This then gives us the slope, m. We can use this to find an estimate of another point on the (true) y curve as follows. The height of the vertical side of the right triangle in the diagram is given by $m*h$, where h is the horizontal distance to the next point. Hence our estimate of the new y value is $\alpha + m*h = y(a) + f(a, y(a))*h$. If we think of y as being a list of y values at (known) x values $x_0=a$, x_1, x_2,…,x_n, we can write the y values as a sequence, $y_0, y_1, y_2,…,y_n$. So we start with given values $x_0=a$, $y_0=\alpha$ and proceed to calculate the following x_i and y_i values using the formula below, until we reach the desired endpoint $x_n=b$.

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[1] As time permits: please be patient.

Eulers Method: Geometric

Summary

To solve the (ordinary[2]) differential equation

(0.1)
$$\frac{dy}{dx} = f(x, y), a \le x \le b, y(a) = y_0$$

we use the iterative formula

(0.2)
$$y_k = y_{k-1} + h * f(x_k, y_k), k = 1, 2, 3, ..., n$$

Note that we must choose a value for h before we can begin. We need h to be small, otherwise the accuracy of our method will be unacceptable.
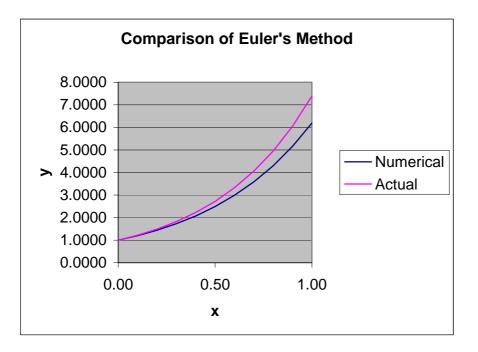
Example

Solve the ODE $\frac{dy}{dx} = 2y; y(0) = 1, 0 \le x \le 1$.

Now we can easily see that the solution to this ODE is $y = e^{2t}$. Hence after trying our numerical procedure we will be able to plot the numerical solution and the actual solution together and compare them. The table below was prepared in Excel, as was the graph following.

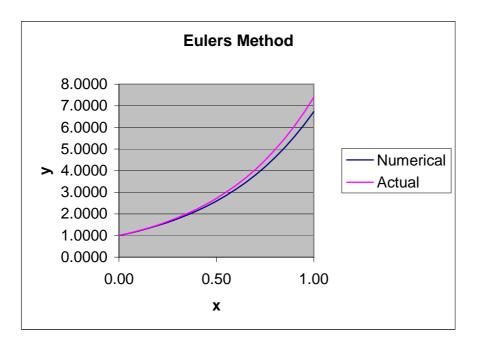| Numerical solution of ODE using Euler's Method				
dy/dx=2y, y(0)=1, 0<=x<=1				
h==>	0.1			
x0==>	0			
y0==>	1			

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[2] Ordinary because it involves no partial derivatives, where we differentiate a function of more than one variable.

xn	yn	f(xn,yn)	h*f(xn,yn)	yn+1	True yn
0.00	1.0000	2.0000	0.2000	1.2000	1.0000
0.10	1.2000	2.4000	0.2400	1.4400	1.2214
0.20	1.4400	2.8800	0.2880	1.7280	1.4918
0.30	1.7280	3.4560	0.3456	2.0736	1.8221
0.40	2.0736	4.1472	0.4147	2.4883	2.2255
0.50	2.4883	4.9766	0.4977	2.9860	2.7183
0.60	2.9860	5.9720	0.5972	3.5832	3.3201
0.70	3.5832	7.1664	0.7166	4.2998	4.0552
0.80	4.2998	8.5996	0.8600	5.1598	4.9530
0.90	5.1598	10.3196	1.0320	6.1917	6.0496
1.00	6.1917	12.3835	1.2383	7.4301	7.3891

Comparison of Euler's Method

The numerical solution is slightly lower than the true solution, because in this case $f'(x)$ is constantly increasing. Hence the error at every step is positive, because as soon as we travel even a slight distance in the x direction at constant slope, we are falling behind the true solution. Let's try a smaller value of h and see how the numerical solution compares.

xn	yn	f(xn,yn)	h*f(xn,yn)	yn+1	True yn
0.00	1.0000	2.0000	0.1000	1.1000	1.0000
0.05	1.1000	2.2000	0.1100	1.2100	1.1052
0.10	1.2100	2.4200	0.1210	1.3310	1.2214
0.15	1.3310	2.6620	0.1331	1.4641	1.3499
0.20	1.4641	2.9282	0.1464	1.6105	1.4918
...					
0.85	5.0545	10.1089	0.5054	5.5599	5.4739
0.90	5.5599	11.1198	0.5560	6.1159	6.0496
0.95	6.1159	12.2318	0.6116	6.7275	6.6859
1.00	6.7275	13.4550	0.6727	7.4002	7.3891

Eulers Method

Using a smaller h value has improved the error at every x value along the way. Often, the derivative will be positive in some places and negative in others, so our numerical estimate will sometimes be above the true value and sometimes below it. Here, the numerical value is always below the true value, which is somewhat unusual.

Let us try another example, more difficult this time. Solve the ODE $\frac{dy}{dx} = \frac{2x+1}{y}$, subject to the condition $y(0)=3$. Because this is a variables separable ODE, we can solve it as follows. (We will then use the true solution to check our numeric solution for accuracy.)

$$\frac{dy}{dx} = \frac{2x+1}{y}$$

$$y\,dy = (2x+1)dx$$

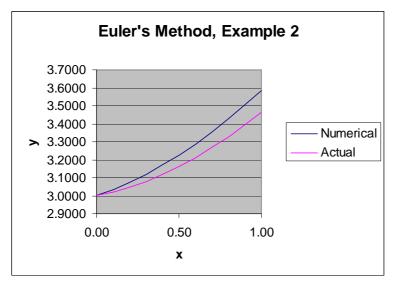$$\int y\,dy = \int (2x+1)dx$$

$$\frac{1}{2}y^2 = x^2 + x + c$$

$$y = \sqrt{2x^2 + 2x + c}$$

At $x = 0$, $y = 3$

$3 = \sqrt{c}$, ie $c = 9$

$$y = \sqrt{2x^2 + 2x + 9}$$

Putting the DE and initial conditions into an Excel spreadsheet, we obtain the following. (Using h=0.1 again, and assuming we wish to know $y(1)$.)

h==>	0.1				
x0==>	0				
y0==>	3				
xn	*yn*	*f(xn,yn)*	*h*f(xn,yn)*	*yn+1*	*True yn*
0.00	3.0000	0.3333	0.0333	3.0333	3.0000
0.10	3.0333	0.3956	0.0396	3.0729	3.0199
0.20	3.0729	0.4556	0.0456	3.1185	3.0463
0.30	3.1185	0.5131	0.0513	3.1698	3.0790
0.40	3.1698	0.5679	0.0568	3.2265	3.1177
0.50	3.2265	0.6199	0.0620	3.2885	3.1623
0.60	3.2885	0.6690	0.0669	3.3554	3.2125
0.70	3.3554	0.7153	0.0715	3.4270	3.2680
0.80	3.4270	0.7587	0.0759	3.5028	3.3287
0.90	3.5028	0.7994	0.0799	3.5828	3.3941
1.00	3.5828	0.8373	0.0837	3.6665	3.4641

Euler's Method, Example 2

Note that on this occasion the numerical solution is always slightly greater than the true solution. In general, we hope that there is no such consistent bias; ie we hope that sometimes our solution is above the true curve and sometimes below it.