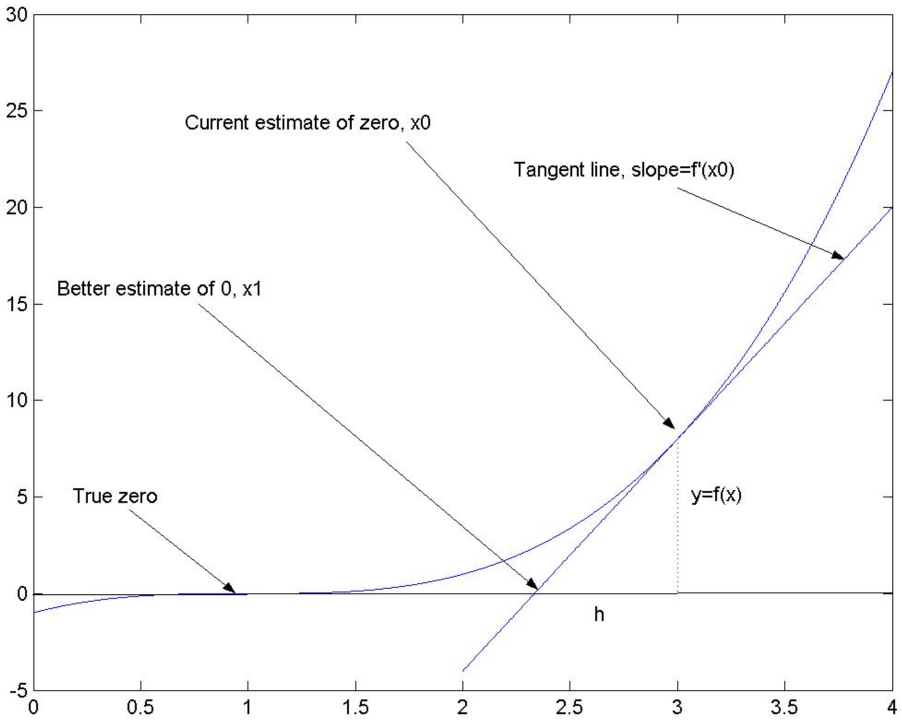
28.4: Approximate Solution Of An Equation – Newton's Method

Frequently it becomes necessary to solve an equation of the form $f(x)=0$. For example, while trying to optimise some function we must find where the derivative is 0. Or we seek the value of x that makes $f(x)=K$, for some constant K. Then $f(x)-K=0$, and we can solve this by Newton's Method.

Consider the diagram below.



We have a current estimate of the root, x_0. This is not a very good estimate, and we seek a better estimate, say x_1. Then one way of seeking this estimate is to extend the tangent line of f at x_0 until it reaches the x axis. This will, in general, be a better estimate than x_0, and we call it x_1. How can we find the location of x_1? Well, if we know the derivative of $f(x)$ then we can find the slope of the tangent line, $f'(x_0)$. We know the height of the function at x_0, of course, $f(x_0)$. So we can find the horizontal distance along the x axis from x_0 to x_1, which we have labelled h in the figure above.

$$f'(x_0) = \frac{y}{h} = \frac{f(x_0)}{h}$$

$$h = \frac{f(x_0)}{f'(x_0)}$$

and hence our new estimate $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. Of course we repeat this method over and over, and so we summarise it in the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. We refer to the repeated use of this formula as iterating the formula.

Example (p887)
Find one root of the equation $x^3 - 5x = 5$. First of all, we rewrite it as follows. $f(x) = x^3 - 5x - 5 = 0$. Note that *f*(1)=-9, while *f*(3)=7, so there must be a zero between 1 and 3. Let us try 2 as our x_0. Note also that $f'(x) = 3x^2 - 5$, so that $x_{n+1} = x_n - \frac{x_n^3 - 5x_n - 5}{3x_n^2 - 5}$. Performing a few iterations in Excel, we see that the process converges quite quickly to an approximate solution *x* = 2.6274. Note that this solution was reached quite early, and we could have done less work and still had just as accurate an answer. In general, a computer package will ask for a tolerance, and once the function value becomes lower than that tolerance, it will immediately return the current approximation.

Example 1 (p887)

x0==> 3

xn	f(xn)	f'(xn)	f(xn)/f'(xn)	new x
3	7	22	0.3181818	2.682
2.6818	0.87894	16.58	0.0530237	2.629
2.6288	0.02247	15.73	0.0014284	2.627
2.6274	1.6E-05	15.71	1.024E-06	2.627
2.6274	8.3E-12	15.71	5.26E-13	2.627
2.6274	0	15.71	0	2.627
2.6274	0	15.71	0	2.627
2.6274	0	15.71	0	2.627
2.6274	0	15.71	0	2.627

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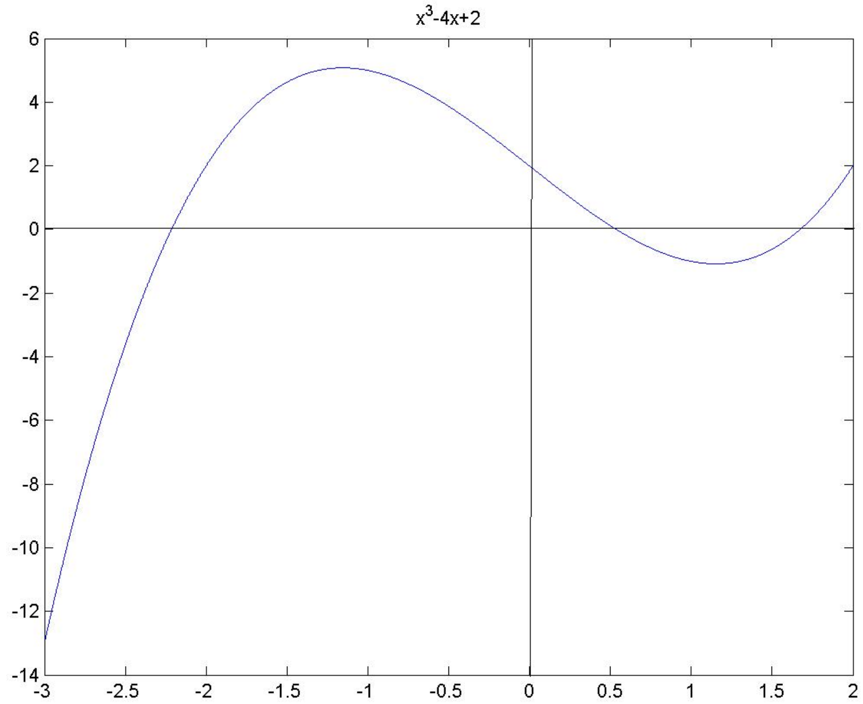
Example

Find all the roots of the equation $x^3 - 4x + 2 = 0$. Firstly, let us sketch a diagram so that we have a good idea of how many roots there are, and approximately where they are.



Hence from the diagram we can see that there should be three zeros, at about x=-2.2, 0.5 and 1.7. We shall use those as starting values for our iterative process.

Example 2

x0==> -2.2

xn	f(xn)	f'(xn)	f(xn)/f'(xn)	new x
-2.2	0.152	10.52	0.0144487	-2.21
-2.2144	-0.0014	10.71	-0.000129	-2.21
-2.2143	-1E-07	10.71	-1.03E-08	-2.21
-2.2143	0	10.71	0	-2.21

Note that using x_0=-2.2 yields an estimated zero of -2.2143. Next, try and find the next zero using x_0=0.5

x0==> 0.5

xn	f(xn)	f'(xn)	f(xn)/f'(xn)	new x
0.5	0.125	-3.25	-0.038462	0.538
0.5385	0.00228	-3.13	-0.000727	0.539
0.5392	8.5E-07	-3.13	-2.73E-07	0.539
0.5392	1.2E-13	-3.13	-3.85E-14	0.539

Once again Newtons method converges rapidly. Finally, the zero near x_0=1.7.

x0==> 1.7

xn	f(xn)	f'(xn)	f(xn)/f'(xn)	new x
1.7	0.113	4.67	0.024197	1.676
1.6758	0.00297	4.425	0.0006716	1.675
1.6751	2.3E-06	4.418	5.132E-07	1.675
1.6751	1.3E-12	4.418	2.997E-13	1.675

What happens if we start further away from the true root? Let us try the above example with x_0=10, say. The example below shows that it takes 9 iterations to come up with an acceptable approximation, instead of the usual 3. Hence it is always worth making the effort to find a good estimate of the root before beginning, if possible.

Example 2

x0==> 10

xn	f(xn)	f'(xn)	f(xn)/f'(xn)	new x
10	962	296	3.25	6.75
6.75	282.547	132.7	2.1294159	4.621
4.6206	82.1662	60.05	1.3683102	3.252
3.2523	23.3911	27.73	0.8434752	2.409
2.4088	6.3414	13.41	0.4729944	1.936
1.9358	1.5109	7.242	0.2086292	1.727
1.7272	0.24369	4.949	0.049237	1.678
1.6779	0.01244	4.446	0.0027982	1.675
1.6751	3.9E-05	4.418	8.916E-06	1.675
1.6751	4E-10	4.418	9.042E-11	1.675
1.6751	0	4.418	0	1.675

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Can Newton's method ever fail us? Yes, as a matter of fact it can. Try to find the zero of $f(x)=\dfrac{1}{1+x^2}-0.5$. The diagram shows us that it does have a zero, and it occurs at about x=1. Let us try Newtons method using x_0=3, say. Note the disastrous results in the spreadsheet excerpt below. What has gone wrong? Note that x_n alternates between positive and negative, and gets larger and larger in absolute value.

Example 3

x0==> 3

xn	f(xn)	f'(xn)	f(xn)/f'(xn)	new x
3	-0.4	-0.06	6.6666667	-3.67
-3.6667	-0.4308	0.035	-12.25589	8.589
8.5892	-0.4866	-0	158.38799	-150
-149.8	-0.5	6E-07	-840358.6	8E+05
840209	-0.5	-0	1.483E+17	#####
-1E+17	-0.5	6E-52	-8.15E+50	8E+50
8E+50	-0.5	-0	1.35E+152	#####
-1E+152	-0.5	#####	#NUM!	#####
#NUM!	#NUM!	#####	#NUM!	#####
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In fact $f'(x)$ gets very close to zero in this case. The side h in our first diagram then overshoots the true zero and ends up too far on the other side of the true zero, further away than our first estimate. This point also has f' very close to zero and hence the next estimate of the root overshoots again, this time back to the original side of the true zero. It continues in this way, getting further and further away from the true zero. Newton's method can be quite vulnerable to this problem, so it always pays to check the derivative of f carefully, and look for good quality starting points.