Mathematical Methods for Engineers 1 (MATH 1063)
Calculus 1 (MATH 1054)

Week 1 Lecture Contents:

Functions, Models and Graphs

1. Functions and Mathematical Modelling — *Edwards and Penney, §1.1*
2. Graphs — *Edwards and Penney, §1.2*
3. Vertical Line Test — *Edwards and Penney, §1.2*

Functions and Mathematical Modelling

Functions are relationships between one variable and other(s). Some simple familiar functions are

- The volume of a sphere in terms of its radius r is

$$V = \frac{\;}{\;}$$

- The volume of a cylinder in terms of its radius r and height h is

$$V = \qquad .$$

- The distance travelled by a falling body dropping from rest after t seconds is

$$s = \frac{\;}{\;}$$

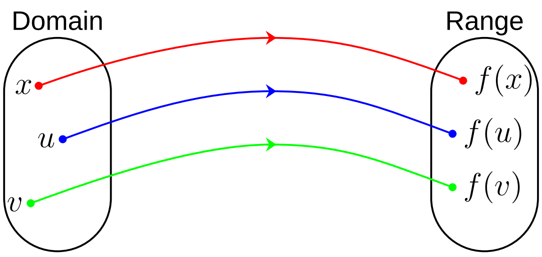
  Here s will be in metres if the gravitational acceleration is $g \approx 9.8$ m/s^2.

Edwards and Penney give the formal definition of a function as

Definition. A real-valued f defined on a set D of real numbers is a that assigns to each number x in D one real number, denoted by $f(x)$.

Definition. The set D of all numbers for which $f(x)$ is defined is called the of the function f and the set of all values $y = f(x)$ is called the of f.

We can represent this pictorially as



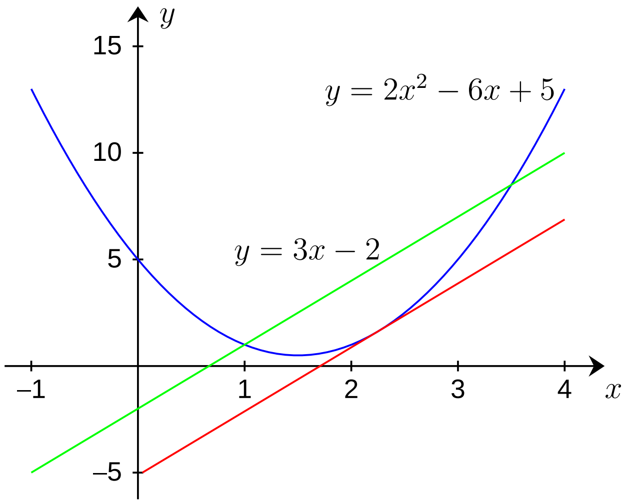
We call x the **variable** and y the **variable** because the value of y depends, through f, on the choice of x.

Example 1. Consider $y = f(x) = 2x^2 - 6x + 5$.

(a) What values can x and y take?

(b) Where does the straight line $y = 3x - 2$ intersect this parabola?

(c) What other straight line, parallel to $y = 3x - 2$, *touches* the parabola?



Solution. (a) The **discriminant** of the quadratic $y = ax^2 + bx + c$ is $\Delta = b^2 - 4ac$.

Firstly, note that the discriminant here is $\Delta = (-6)^2 - 4 \times 2 \times 5 =$, and hence there are zeros of f.

Complete the square:

$$\begin{aligned} f(x) &= \qquad\qquad + 5 \\ &= \qquad\qquad\qquad + 5 \\ &= \end{aligned}$$

Hence the *minimum* value of $f(x)$ is 0.5 when $x = 1.5$.

The function is defined for all real values of x. We can say that the domain is ( ) or $\mathbb{R}$.

The values that y can take are y . We can say that the range is [ ).

(b) $2x^2 - 6x + 5 =$

$= 0$

$(x-1)(\quad) = 0 \Rightarrow x =$

They intersect at $(1, 1)$ and ( ).

(c) Let the straight line be $y =$. This intersects $y = 2x^2 - 6x + 5$ whenever

$$\begin{aligned} 2x^2 - 6x + 5 &= \\ 2x^2 - 9x + (\quad) &= 0. \end{aligned}$$

To "touch" we must have a root which is the equivalent to the discriminant being

$$81 - 8(\quad) = 0 \Rightarrow b = \text{—}$$

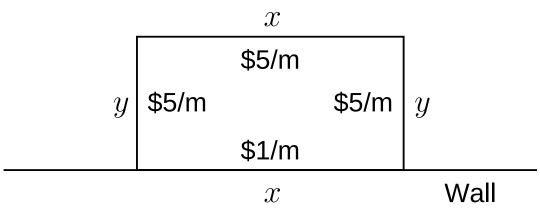
$$\therefore y = 3x \quad \text{— touches the parabola } y = 2x^2 - 6x + 5.$$

Exercise. Find the coordinates of the point where they touch.

(Answer: $x =$ –, $y =$ –)

Example 2. The Animal Pen Problem

The problem is to build a rectangular animal pen using an existing wall as one side, as shown in the diagram below. The fencing material costs \$5 per metre and the wall needs painting at a cost of \$1 per metre. If there is \$180 available, what is the maximum area that can be enclosed.



Solution. The area of the pen is a function of the variables, the length x and the width y:

$$A = f(x, y) = xy.$$

The cost of constructing the pen is

$$C = 180 = \qquad\qquad = \qquad .$$

Hence we can find y as a function of x, namely

$$y = g(x) = \text{—} \qquad\qquad = \text{–} \qquad .$$

Using this, we can eliminate y from the area equation to obtain

$$A(x) = \text{–} \qquad\qquad .$$

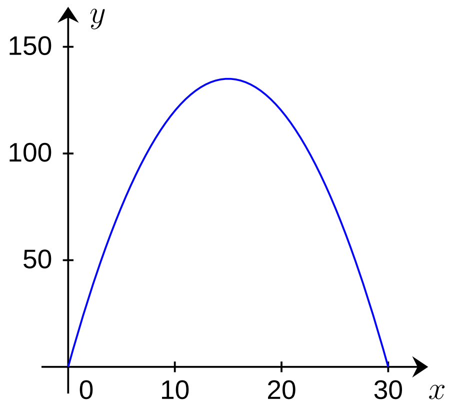
If $x = 0$ we have a degenerate rectangle of base zero, height 18 and area. If $x = 30$ we have a degenerate rectangle of base 30, height 0 and zero area. Thus the complete definition of the area function is

$$A(x) = \text{–} \qquad\qquad , \quad 0 \le x \le 30.$$

We can tabulate some area values

x	0	5	10	15	20	25	30
$A(x)$	0	75	120	135	120	75	0

or plot the graph of the quadratic as shown below.



It would appear from the table, or the symmetry of the quadratic about its maximum, that the maximum area is 135 m^2 when $x = 15$. This can be shown by completing the square on the quadratic.

$$A(x) = -$$
$$= \quad -$$
$$= \quad -$$

Hence the maximum area is m^2 when $x = 15$ m and $y =$ m.

Example 3. Find the domain and range of the function $y = \sqrt{x+1}$.

We are dealing with *real* functions, so square roots of negative numbers are not permitted. Hence $x \geq -1$, and the domain is [ ) and the range is [ ).

Example 4. Find the domain and range of the function $y = \dfrac{1}{\sqrt{x+1}}$.

Additionally to the previous example, division by zero is not permitted. Hence $x > -1$, and the domain is ( ) and the range is ( ).

Intervals

In denoting domains and ranges, sometimes round brackets have been used and sometimes square brackets. An **interval**, such as $(1, 3)$, is one where the endpoints (1 and 3) are included; a **interval**, such as $[-1, 2]$, is one where the endpoints (-1 and 2) included. Half-open intervals such as $[-3, 5)$ or $(-5, -3]$ are possible. Unbounded intervals [ ) or ( ) are open at the ∞ or $-\infty$ end.

Straight Line

Refer to the diagram below.

Slope: $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$

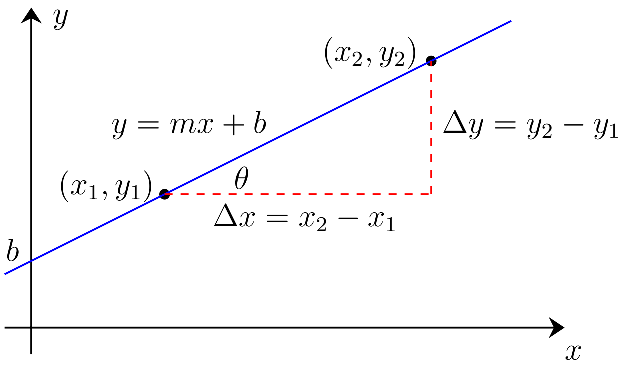
Equation: $= m($ $)$ or $y = mx + b$

where b is the .

If the line is *horizontal*, then $m =$ and if the line is *vertical*, then m is .

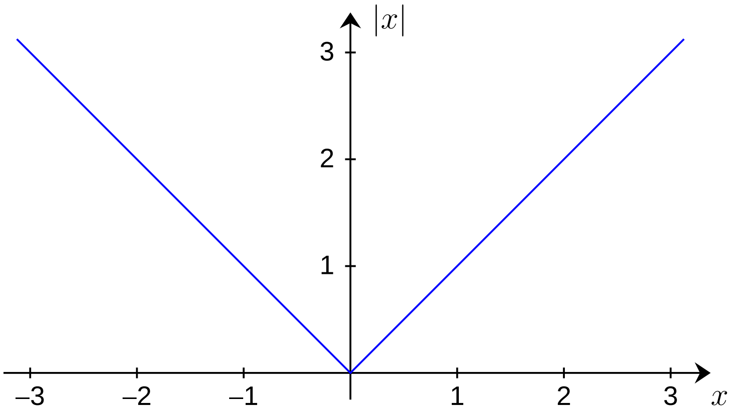
If a line of slope m_1 is perpendicular to another line of slope m_2, then $m_2 = \frac{\quad}{\quad}$.

If θ is the *angle of inclination* to the positive x-axis, then $\tan\theta =$.

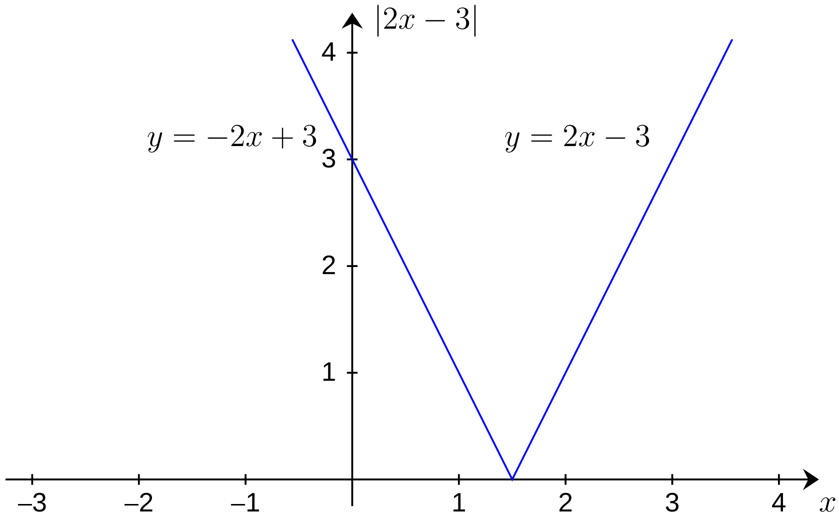


The Absolute Value Function

$$y = |x| = \begin{cases} x, & \text{for } x \geq 0, \\ \quad , & \text{for } x < 0. \end{cases}$$



$$y = |2x - 3| = \begin{cases} \qquad , & \text{for } x \geq -, \\ & \text{for } x < -. \end{cases}$$



Note that the function can be constructed from the two individual functions, or by translating the function $y =$ units to the right, i.e. to the point where $| \quad |$ is .

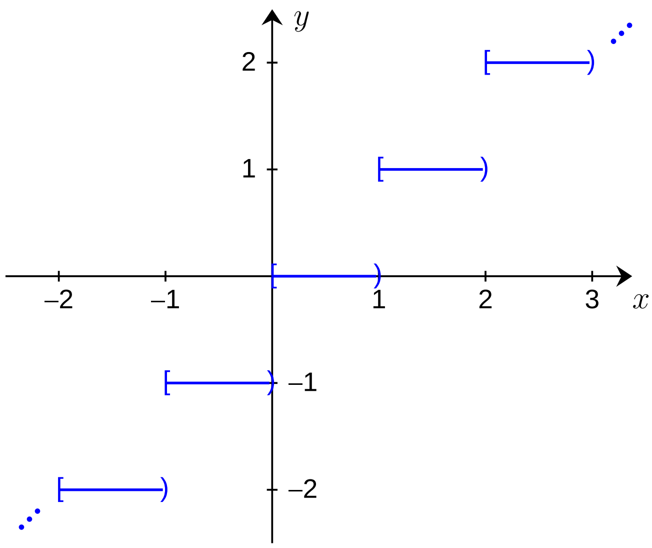
The Floor and Ceiling Functions

The *floor* of x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x. The *ceiling* of x, denoted $\lceil x \rceil$, is the least integer greater than or equal to x.

For example

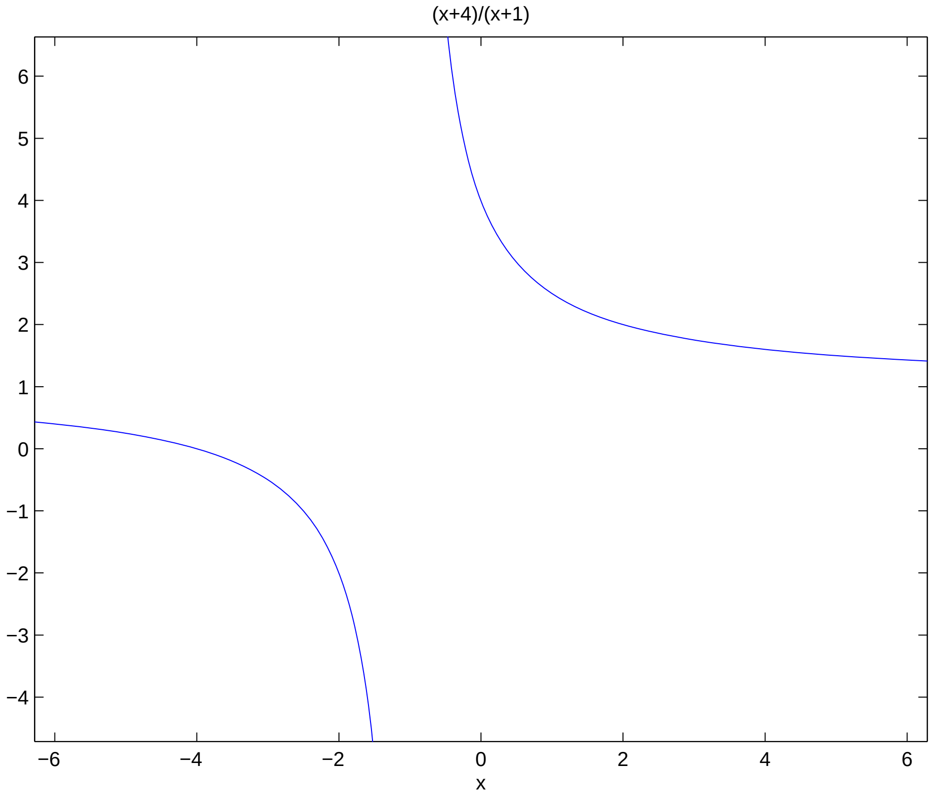
$$\lfloor 7.4 \rfloor = \quad , \qquad \lceil 7.4 \rceil =$$
$$\lfloor -8.7 \rfloor = \quad , \qquad \lceil -8.7 \rceil =$$
$$\lfloor 5 \rfloor = \quad , \qquad \lceil 5 \rceil =$$

The graph of the floor function is shown below.



Example 5. The function $y = \dfrac{x+4}{x+1} = \qquad \underline{\qquad}$.

We can use the **ezplot** command in MATLAB to view the graph of the function. Typing `ezplot('(x+4)/(x+1)')` gives



However, as Jensen says in the preface to "Using MATLAB with Calculus", "one can plot the graph of a function f with the symbolic toolbox's ezplot command without any understanding of what the graph of a function is". Calculating some (x, y) values for the function gives

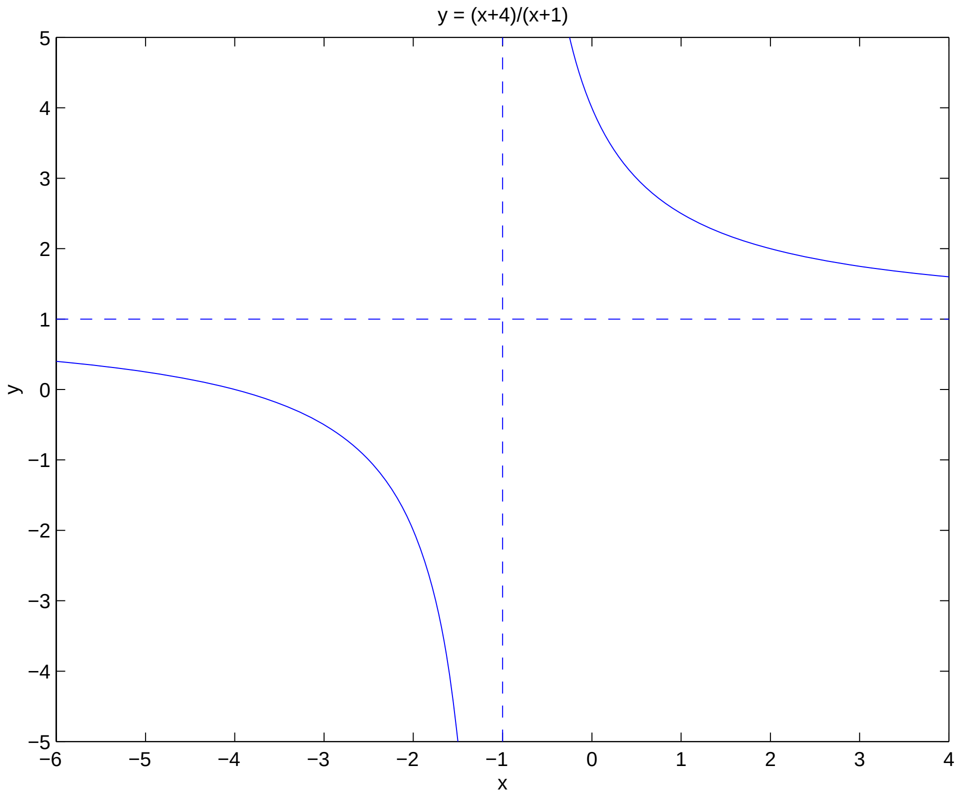
x	-6	-5	-4	-3	-2	-1	0	1	2	3	4
y	0.4	0.25	0	-0.5	-2	$-$	4	2.5	2	1.75	1.6

It is apparent that there is a $\qquad\qquad$ asymptote at $x = -1$, and there is also a asymptote at $y = 1$. This is better seen if we calculate x as a function of y.

$$\begin{aligned} y &= \underline{\qquad} \\ y(x+1) &= \\ x &= \underline{\qquad} \\ &= \underline{\qquad} \end{aligned}$$

If we wished to plot for a y range from -5 to 5, then we would need to plot the graph for $-6 \leq x \leq -1.5$ and $-0.25 \leq x \leq 4$. The following MATLAB m-file will produce the graph shown below.

```
% M-file to plot y = (x+4)/(x+1)
clear all
x1 = -6:0.01:-1.5;
% or x1 = linspace(-6,-1.5,451);
x2 = -0.25:0.01:4;
y1 = (x1+4)./(x1+1);
y2 = (x2+4)./(x2+1);
plot(x1,y1,'b',x2,y2,'b')
hold on
% Plot the asymptotes as dashed lines
x3 = [-1 -1];
y3 = [-5 5];
x4 = [-6 4];
y4 = [1 1];
plot(x3,y3,'--')
plot(x4,y4,'--')
xlabel('x')
ylabel('y')
title('y = (x+4)/(x+1)')
```



Graphs

We have seen the graphs of many types of functions so far. Edwards and Penney give the formal definition of the graph of a function as

The **graph** of the function f is the graph of the equation .

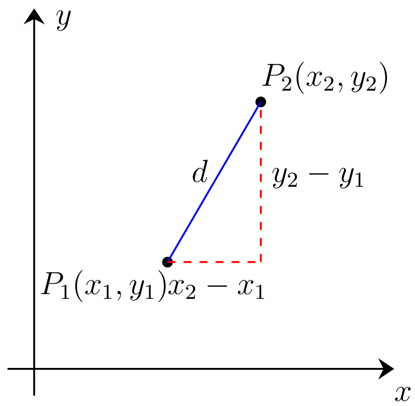
This is a specific form of the more general definition of the graph of an equation, which is

The **graph** of an equation in two variables x and y is the set of all points (x, y) in the plane that satisfy the equation.

For example, the Pythagorean theorem implies the **distance formula**

$$d = \sqrt{(x2 - x1)^2 + (y2 - y1)^2},$$

as shown in the figure below.

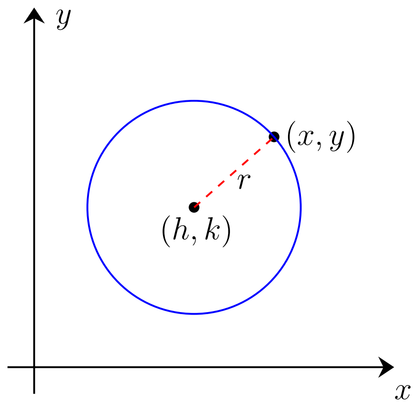


This distance formula tells us that the graph of the equation

is a circle of radius r and centre at the origin $(0, 0)$. The more general equation

$$(\quad)^2 + (\quad)^2 = r^2$$

is a circle of radius r and centre at the point ( ), as shown in the figure below.



The circle is the graph of a function. This is supported by the Vertical Line Test which is described below.

The Vertical Line Test

Each line through a point in the domain of a function meets its graph in one point.

Note that the top half of the circle $x^2 + y^2 = r^2$ has the equation

$$y = \overline{}, \qquad -r \leq x \leq r,$$

and is a function.

Example 6. The equation

$$x^2 + y^2 - 6x - 8y - 75 = 0$$

can have the square completed in the x and y terms to obtain

$$(\qquad)^2 + (\qquad)^2 = \qquad ,$$

which is a circle of radius 10 and centre at ( ).

Example 7. More exotic graphs can be formed from equations in x and y. For example, see Figure 1.2.5 on page 13 of Edwards and Penney. Another example is the **cardioid** whose equation is

$$(x^2 + y^2 - x)^2 = x^2 + y^2,$$

and whose graph appears below.