Mathematical Methods for Engineers 1 (MATH 1063) Calculus 1 (MATH 1054)

Week 2 Lecture Contents

Functions and Limit

Polynomials and Rational Functions

A **polynomial** of **degree** n is a function of the form

$$p(x) = \qquad + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x +$$

where the coefficients $a_0, a_1, \ldots, a_n$ are fixed   numbers and   . Thus an n th-degree polynomial is a sum of constant multiples of the **power functions**

$$1, \ x, \ x^2, \ \ldots, \ x^{n-1}, \ x^n.$$

A first-degree polynomial is a *linear function*   whose graph is a straight line, and a second-degree polynomial is a   *function.*

The x-intercepts of the graph of a polynomial are the   of the polynomial, $y = p(x) = 0$. An n th-degree polynomial will have n zeros, some possibly   and some possibly   . Hence an n th-degree polynomial will have at most n real zeros.

Example 1. The MATLAB produced graph of the cubic polynomial $y = x^3 - 3x^2 + 2$ is shown below. We can see that this polynomial has 3 real zeros. However the cubic $y = x^3 - 3x^2 + 5$ would be this polynomial vertically translated by   units, and would only have   real zero.

$$y = x^3 - 3x^2 + 2$$
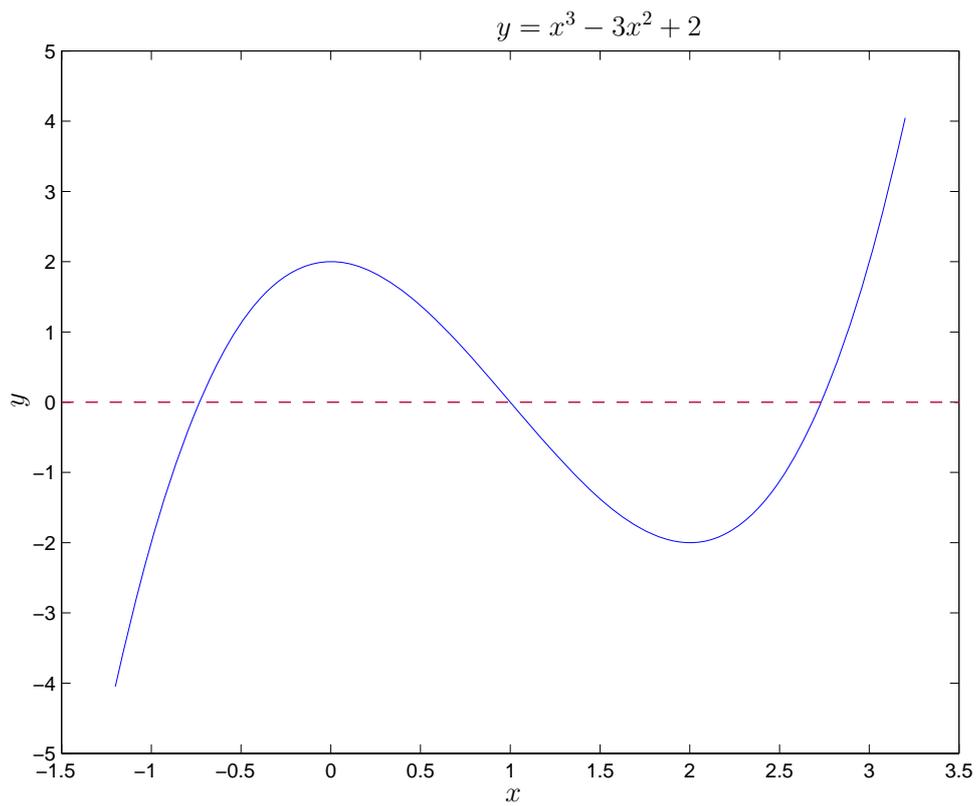
A _____ **function** is a quotient of two polynomials $p(x)$ and $q(x)$, i.e.

$$f(x) = \frac{\qquad}{\qquad}.$$

We have seen an example of a rational function in Example 5.

Composite Functions

Functions can be added, subtracted, multiplied and divided (provided that the denominator is not zero). An additional operation is composition of functions. The composite function $f \circ g$ is defined as

$$(f \circ g)(x) = \qquad .$$

The *domain* of $f \circ g = \{x : x \in \text{domain of } g \qquad g(x) \in \text{domain of } f\}$.

Question. Are $\sqrt{x^2}$, $(\sqrt{x})^2$ and x all the same?

NO!

$$
\begin{aligned}
x : &\qquad \text{Domain } \mathbb{R}, \text{ Range } \mathbb{R} \\
\sqrt{x^2} : &\qquad \text{Domain } \mathbb{R}, \text{ Range } \mathbb{R}^+ \\
(\sqrt{x})^2 : &\qquad \text{Domain } \mathbb{R}^+, \text{ Range } \mathbb{R}^+.
\end{aligned}
$$

If $f(x) = x^2$ and $g(x) = \sqrt{x}$, then

$$\sqrt{x^2} = g(f(x)) = (g \circ f)(x),$$

whereas

$$(\sqrt{x})^2 = f(g(x)) = (f \circ g)(x).$$

Example 2. Let $f(x)$ and $g(x)$ be

$$
\begin{aligned}
f(x) = x^2 - 1 : &\qquad \text{Domain } \{x : \qquad \} \\
g(x) = \frac{1}{x} : &\qquad \text{Domain } \{x : x \in \mathbb{R} \qquad \}
\end{aligned}
$$

What are the composite functions $f(g(x))$ and $g(f(x))$, and what are their domains?

$$(f \circ g)(x) = \text{---} \qquad : \qquad \text{Domain } \{x : x \in \mathbb{R} - \{0\}\}$$

$$(g \circ f)(x) = \text{------} \qquad : \qquad \text{Domain } \{x : x \in \mathbb{R} - \{\quad\}\}$$

The Tangent Line Problem

The computation technique of "calculus" has its origins in two geometrical problems. The second, to be studied later in the semester, is the "area under a curve" problem. The first is the "tangent line to a curve" problem, where "tangent" means "touching".

If we refer to the diagram below, the line L is $\qquad$ to the curve $y = f(x)$ at the point P, $(a, f(a))$. The line K is a $\qquad$ line, and intersects the curve at P and at Q, $(x, f(x))$. If we imagine that the point Q is moving down the curve towards the point P, then the line K is approaching L, and *the slope of* K *is approaching the slope of* L. In the limiting case, as Q becomes P, the slopes of the secant and tangent lines will be the same.

We have

$$\text{Slope of K} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

and hence

$$\text{Slope of L} = m = \lim_{x \to a} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$
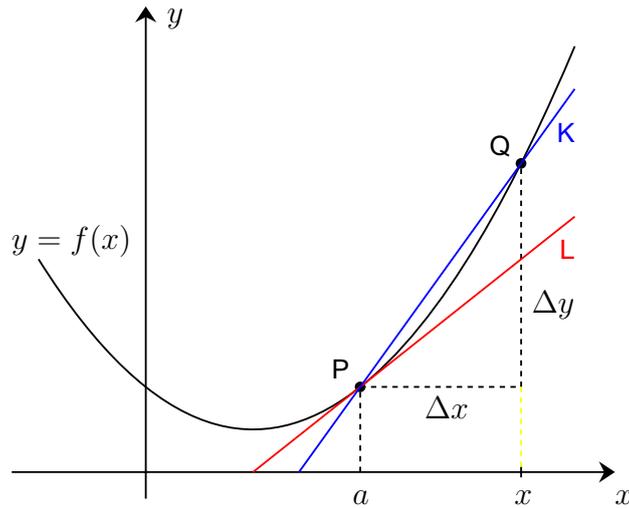
An alternative common notation is to consider the point Q to be at $(a + h, f(a + h))$, so that

$$\text{Slope of L} = m = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Two examples of the practical uses of the slope of a curve are as follows.

Example 3. Suppose that $y = f(t)$ represents the distance, in kilometres, travelled by a car at time t in hours. Then the slope of the line tangent to the curve $y = f(t)$ at the point $(t, f(t))$ is the *instantaneous* ⎵⎵⎵⎵ in km/h of the car at time t.

Example 4. Suppose that $P(t)$ is the population of a species, in individuals, at a time t in years. Then the slope of the line tangent to the curve $y = P(t)$ at the point $(t, P(t))$ is the *rate of growth* of the population in individuals per year at time t.

If $V(t)$ is the value of an asset, in dollars, at a time t in years, what does the slope of the curve $y = V(t)$ represent?

Limit

The determination of the slope of a tangent line to a function $y = f(x)$ leads to the limit concept. Edwards and Penney state

Idea of the Limit

We say that the number L is the **limit** of $f(x)$ as x approaches a (from either side of a) provided that we can make the number $f(x)$ as close to L as we please merely by choosing x sufficiently near, though not ⎵⎵⎵⎵ to, the number a.

Example 5. $\displaystyle \lim_{x \to 2} \frac{x^2 + 3}{x - 3}$

We can see that as ⎵⎵⎵⎵ the numerator approaches ⎵⎵⎵ and the denominator approaches ⎵⎵⎵ , so the limit is $\dfrac{7}{-1} =$ ⎵⎵⎵ .

Suppose this is changed to $\displaystyle \lim_{x \to 3} \frac{x^2 + 3}{x - 3}$.

Now the limit does not exist (DNE) because substituting $x = 3$ gives $\dfrac{12}{}$.

We could take a **right-hand limit** where x approaches from above

$$\lim_{x \to 3^+} \frac{x^2 + 3}{x - 3} = \qquad,$$

or a **left-hand limit** where x approaches from below

$$\lim_{x \to 3^-} \frac{x^2 + 3}{x - 3} = \qquad.$$

However

$$\lim_{x \to 3} \frac{x^2 + 3}{|x - 3|} =$$

regardless of whether the limit is left- or right-hand, as the function is always positive.

Refer to Edwards and Penney on page 79 for one-sided limits and page 82 for infinite limits.

Example 6. Another example of one-sided limits being different is for the floor function $\lfloor x \rfloor$, where the left- and right-hand limits are different at integer values of x. For instance

$$\lim_{x \to 2^+} \lfloor x \rfloor = \qquad \text{but} \qquad \lim_{x \to 2^-} \lfloor x \rfloor = \quad.$$
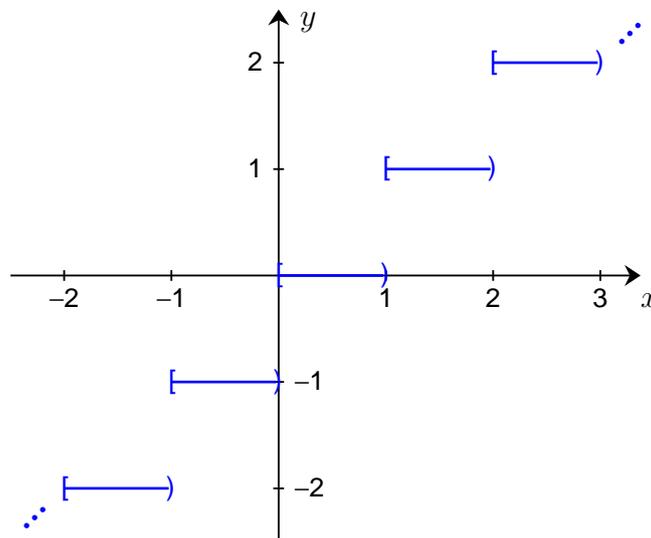
Example 7. $\displaystyle \lim_{x \to 1} \frac{x^3 - 2x + 1}{x^2 - 1}$

If we substitute $x = 1$ into the function, then the result is $-$ which is undefined. However if we take values of x getting closer and closer to 1 from both above and below, then the following is obtained.

x	$\dfrac{x^3 - 2x + 1}{x^2 - 1}$	x	$\dfrac{x^3 - 2x + 1}{x^2 - 1}$
0	-1	2	1.6667
0.5	-0.1667	1.5	1.1
0.9	0.3737	1.1	0.6238
0.99	0.4875	1.01	0.5125
0.999	0.4987	1.001	0.5012
0.9999	0.4999	1.0001	0.50001

It would appear that the limit is 0.5. The limit is equivalent to

$$\lim_{x \to 1} \frac{(x-1)(x^2 + x - 1)}{x^2 - 1} = \lim_{x \to 1} \frac{x^2 + x - 1}{} = -\qquad .$$

Example 8. $\lim\limits_{x \to 0} (1 + x)^{1/x} = e$

This is discussed further in the section on exponential and logarithmic functions.

Example 9. What is $\lim\limits_{x \to \infty} (\sqrt{x^2 + x} - x)$? Is it 0, ∞ or something else?

There are several ways to analyze this. One is to multiply

$$(\sqrt{x^2 + x} - x) \times \frac{(\overline{\qquad})}{(\overline{\qquad})}$$

and simplify. Try this as an exercise.

Another is to expand in a series.

$$(\sqrt{x^2 + x} - x) = x \left(\overline{\qquad} \right)$$

and

$$\left(1 + \frac{1}{x}\right)^n = \qquad - \ \frac{\qquad}{\qquad} + \text{terms in } \frac{1}{x^3} \text{ etc.}$$

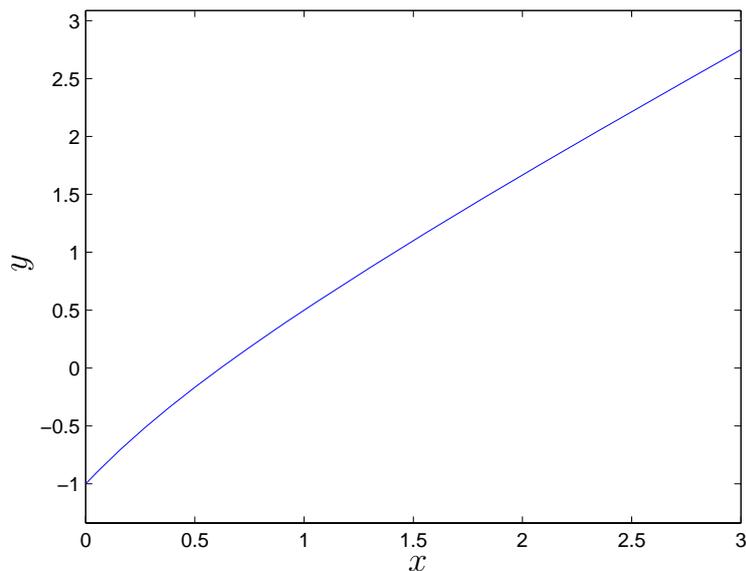**Zooming to Find a Limit**

The MATLAB *zoom* function turns on zooming in and out of a 2-D plot. Clicking the left-hand mouse button will zoom in on the cursor region, and clicking the right-hand mouse button will zoom out.
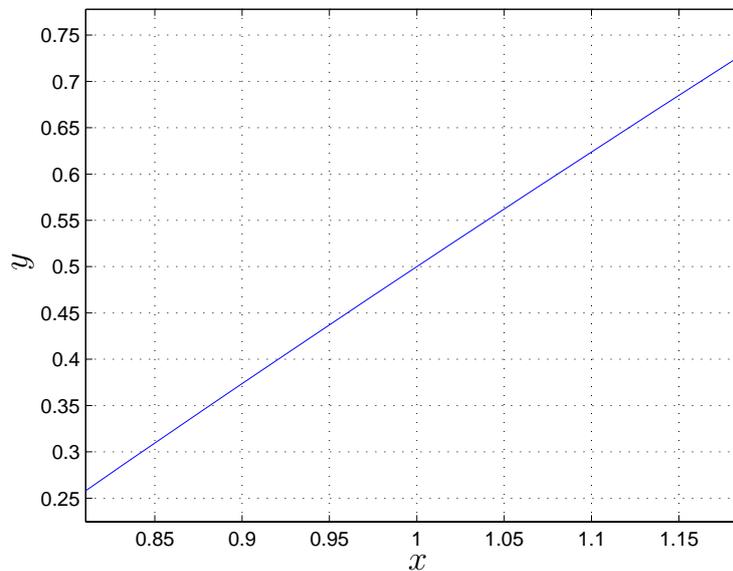
We can plot part of the graph of the function $y = \dfrac{x^3 - 2x + 1}{x^2 - 1}$ in MATLAB using *ezplot*:

```
ezplot('(x^3 - 2*x + 1)/(x^2 - 1)',[0,3])
```

This gives the graph:

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Although the function is not defined at $x = 1$, MATLAB 's version 7, Release R2007b, puts a "patch" at $x = 1$, not displaying a "hole" at $x = 1$. Let's enter the command, `grid on`, and zoom in three times, to obtain the following.

It "looks like" the limit of 0.5 at $x = 1$ is being approached "linearly" from both sides.

Limit Laws

Edwards and Penney list a number of Limit Laws on pages 68–71. One example is the following.

Product Law

If both of the limits

$$\lim_{x \to a} f(x) = L \qquad \text{and} \qquad \lim_{x \to a} g(x) = M$$

exist, then

$$\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] = \quad .$$

That is, the limit of the products is the product of the limits.

Similar laws apply for the addition and subtraction of limits, quotients of limits and composite limits. Essentially, the limit of the sum of functions is equal to the sum of the individual limits, provided that the limits exist. Similar statements apply for the other limit laws.