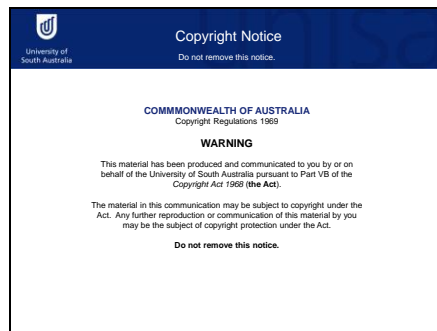
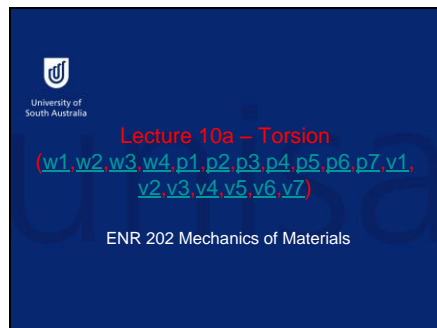


ENR202 Mechanics of Materials Lecture 10A Slides and Notes

Slide 1

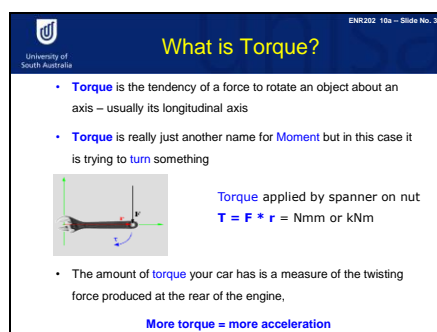


Slide 2



Hello and welcome to Lecture Summary 10a. In this lecture summary, we will be looking at torsion. Click on the live links for further information.


Slide 3



First of all, we need to understand torque. Torque is the tendency of a force to rotate an object about its longitudinal axis. For example, if you want rotate a bolt to fit something, the spanner rotates the bolt by applying force at the end of the spanner, as shown in the image here. In another example, the amount of torque in your car is a measure of the twisting force produced at the rear of the engine. All this means that torque is a 'moment'. As you know, moment is force multiplied by distance. As you can see in this slide, F is the force and 'r' is the distance between the centre of the bolt to where the force is applied. The units of torque are the same as the

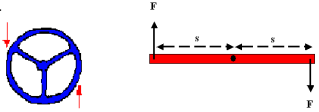
units of moment: that is, Newton mm or Kilo Newton meters.

Slide 4

 **Force Couples** ENR202 10a - Slide No. 4

A special case of moments is called a **couple**.

A **couple** consists of two **parallel** forces that are **equal** in magnitude, **opposite** in **sense** and **do not** share a line of action.



A **couple** does not produce any translation, only **rotation**. The **resultant** force of a couple is **zero**. BUT, the resultant of a couple is not zero; it is a **pure moment (or torque)**.

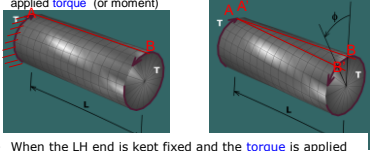
We also need to understand force couples. A force couple is a special case of moment. This couple consists of two equal parallel forces acting in an opposite direction, as shown in the figure on the right. For example, we have wheels that have two equal forces acting in parallel and opposite directions.

Here the torque force 'F' multiplied by the distance is 's' plus 's' is '2s'. All forces in a vertical direction are equal to zero. That means we don't have resultant force. However, the moment is not zero. If we don't have normal force, we don't have any translational displacement. If moment is not zero, then we have rotational displacement. Therefore, force couples produce only rotational displacement without any translational displacement. This is also known as pure moment.

Slide 5

Torsion

- In solid mechanics, **torsion** is the twisting of an object due to an applied **torque** (or moment)
- When the LH end is kept fixed and the **torque** is applied anti-clockwise to the cylinder, the RH end will rotate w.r.t. the LH end.
- This angle of rotation is known as the **angle of twist ϕ**



In solid mechanics, torsion is the twisting of an object due to an applied torque or moment. This produces torsional deformation.

For example, a circular cylinder has a fixed left hand end, and torque is applied in an anti-clockwise direction at the right hand end, as shown in the left figure. If you apply torque in an anti-clock wise direction at the right hand end, the reaction torque at the fixed end is in a clockwise direction. In other words, the opposite direction of torque is applied, as shown in the figure. We choose two points A and B, where Point A is at the fixed end, as shown in the left figure. Before torque is applied, we have the A point at the left hand end and the B point at the right hand end, as shown in the left figure. Once we apply the torque at the right hand end, point A won't move because of the fixed end, and point B will move to B dash at the right hand end, as shown in the right figure.

We already know that torque produces only rotational displacement, not translational displacement. The rotational displacement is 'PI' as shown in the right figure at the right hand end. This PI is just an angle of twist, also known as an angle of rotation. We have zero rotational displacement at the fixed end and larger rotational displacement at the right hand end. We have linear variation from the left hand end to the right hand end after the torque is applied. That means that a larger length of cylinder will produce larger rotational displacement.

Slide 6

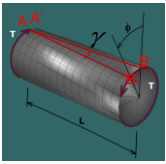
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ENR202 10a - Slide No. 6

Angle of twist ϕ

The angle of twist ϕ changes along the length of the bar, L .

For the prismatic cylinder shown, the angle of twist ϕ varies linearly from zero at the LH end to $\max \phi$ at the RH end.



As long as the total angle of twist ϕ of the bar is small, neither the length of the bar nor its radius will change.

The angle of twist changes along the length of the bar. We have zero angle of twist at the fixed end and we have maximum angle of twist at the free end where the torque is applied, as shown in the figure, and we have linear variation. The angle of twist is related to the centre of the cylinder. However, what about the angle between the AB line and the A dash B dash line? Of course, A and A dash points are the same, because we have zero angle of twist at the fixed end. This angle is called torsional shear strain, denoted as gamma. Therefore, we have maximum torsional shear strain at the circumference of the cylinder, and zero torsional shear strain at the centre of the cylinder, and linear variation of torsional shear strain from the centre to the circumference.

We have two symbols: one is the angle of twist, ϕ , and the other is the torsional shear strain, gamma, as shown in the figure.

Slide 7

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Torsional Shear Strain

ENR202 10a - Slide No. 7

- If we look closely at the squares on the outside of the cylinder, although they stay flat, we can see that they must actually deform slightly (strain)




Figure 1

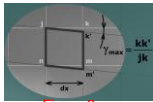


Figure 2

- The RHS of the square rotates w.r.t. the LHS by a small angle γ which is proportional to the angle of twist ϕ at that point along the length
- γ represents the torsional shear strain of the square

If we look closely at the squares on the circumference of the cylinder as shown in the figure 2, we see that lines jk and nm are prior to the torque, and jk' and nm' are after the torque. The right hand side of the square rotates with respect to the left hand side of square by a small angle, γ , which is proportional to the angle of twist ϕ at that point along the length. The torsional shear strain is equal to kk' divided by jk .

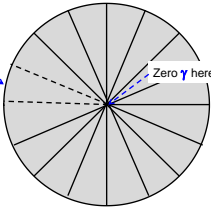
Slide 8

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Torsional Strain

ENR202 10a - Slide No. 8

- If we look at a cross section of the cylinder, it is clear that the maximum shear strain must occur at the outside edge of the cylinder.



Max γ here

Zero γ here

- If the maximum shear strain occurs at the surface, then it follows that the strain reduces linearly as we go closer to the centre (zero)

Therefore, we have maximum shear strain occurring at the circumference of the cylindrical cross section and zero shear strain at the centre of cylindrical cross section, and the shear strain follows linear variation from centre to circumference. The torsional shear strain is directly proportional to the angle of twist, so if the angle of twist increases, then the torsional shear strain also increases.

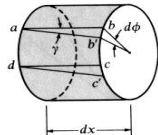
Slide 9

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Torsional Strain

ENR202 10a - Slide No. 9

The torsional shear strain at any point, a distance, r , from the centre is calculated by:

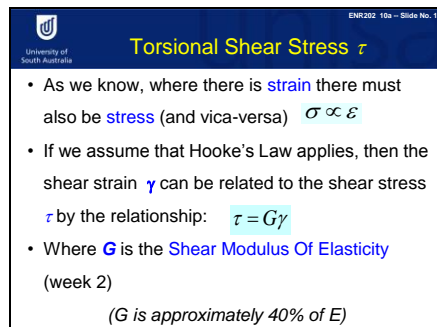
$$\gamma = r \frac{d\phi}{dx}$$


Note that the preceding equations are based only upon geometric concepts and are valid for a circular bar of any material, whether elastic or inelastic, linear or nonlinear.

We can calculate torsional shear strain at any point of the cross section. Consider the small segment of length 'dx', as shown in the figure, and we have angle of twist 'd ϕ ' at the center of the cylinder. Now we can calculate the distance between b and b' as equal to r times 'd ϕ ', where r is the radius from the center to the point where you need to calculate the torsional shear strain. We can calculate the distance between b and b' with respect to the length of the segment: ' γ ' times 'dx'. Therefore ' γ ' times 'dx' is equal to ' r ' times 'd ϕ '. We will get torsional shear strain equal to

'r' times 'd FI' divided by 'dx' .

Slide 10



ENR202 10a - Slide No. 10

Torsional Shear Stress τ

- As we know, where there is **strain** there must also be **stress** (and vica-versa) $\sigma \propto \epsilon$
- If we assume that Hooke's Law applies, then the shear strain γ can be related to the shear stress τ by the relationship: $\tau = G\gamma$
- Where **G** is the **Shear Modulus Of Elasticity** (week 2)
(G is approximately 40% of E)

We already know that where there is strain there must be stress: stress is proportional to strain. This is called Hooke's law. We looked at this in lecture 2a, slide 15. What about shear stress to shear strain? If we assume that Hooke's law applies in shear stress to shear strain, the constant of proportionality of shear stress to shear strain is named the modulus of rigidity.

As we studied in lecture 2a, slide 21, there is a relationship between Young's modulus of material and the modulus of rigidity in terms of Poisson's ratio. The modulus of rigidity is denoted as 'G' and Young's modulus of material is denoted as 'E'. G is approximately equal to 40% of E.

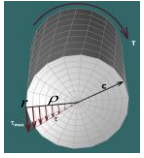
Slide 11

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Torsional Shear Stress τ

ENR202 10a - Slide No. 11

- Like the shear strain, the maximum shear stress τ also acts on the extreme fibres of the cylinder (like beams)


$$\frac{\tau_{\max}}{r} = \frac{\tau}{\rho}$$

- The shear stress τ also varies linearly from a maximum at the extreme fibres to zero in the centre

We already know that maximum torsional shear strain occurs at the circumference and zero torsional shear strain at the centre. If we have shear strain, we have shear stress. Therefore, maximum shear stress occurs at the circumference and zero at the centre, and follows linear variation as shown in the figure. The maximum shear stress occurring at the circumference is denoted as 'tau max' and 'tau' is the shear stress at any point between the centre and the circumference. 'r' is the radius of the cylinder, 'rho' is the distance from the centre to any point between the centre and the circumference. Therefore, shear stress at any point from the centre to the circumference is equal to maximum shear stress times 'rho' divided by the radius of the cylinder, as shown in the equation.

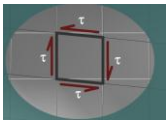
Slide 12

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Pure Shear

ENR202 10a - Slide No. 12

- When a volume element of material only carries shear, the material is subjected to *pure shear*
- Under pure shear, all four shear stresses must have equal magnitude and be directed either toward or away from each other at opposite edges of the element.



- Even though we have represented the shear stress as acting on the edges of the square, in reality the shear stress actually acts on an entire surface

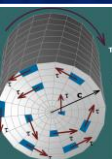
The volume element carries only shear at each side of the rectangle, as shown in the figure. We called this pure shear. Under pure shear, all four shear stresses must have equal magnitude and be directed either toward or away from each other at opposite edges of the element, as shown in the figure. Even though we have represented the shear stress as acting on the edges of the square, in reality the shear stress is acting on an entire surface.

Slide 13

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The Torsion Formula

The relationship between the applied torque T and the shear stress τ may be determined from the condition that the resultant couple of the shear stresses acting over the entire cross section must be statically equivalent to the applied torque, therefore:

$$\frac{\tau_{\max}}{r} = \frac{\tau}{\rho}$$


$$T = \int dM = \int \rho \tau dA = \int \rho \left(\frac{\tau_{\max}}{r} \rho \right) dA = \frac{\tau_{\max}}{r} \int \rho^2 dA = \frac{\tau_{\max}}{r} J \Rightarrow \tau_{\max} = \frac{Tr}{J}$$

in which $J = \int \rho^2 dA$ is the polar moment of inertia of the circular cross section.

Consider the infinitesimal segment as shown in the figure. The shear stress direction is the same as the torque applied direction. The torque is also one of moment. Integration of that moment over a small elemental area is torque. 'tau' is the shear stress acting on a small element, 'rho' is the distance between the center to the small element and 'dA' is the area of the small element. The moment over a small element is equal to the force multiplied by the distance r. The force is shear stress multiplied by the area of the small element 'dA'. We have already derived the equation for calculation of tau over an small element; it is maximum shear stress times 'rho' divided by r. 'tau max' and 'r' are constant values for this particular cross section and torque. The integration of rho squared 'dA' is just the polar moment of inertia (we studied polar moment of inertia in lecture 5a, slide number 14). Here, J is the polar moment of inertia of the circular cross section. Finally we get the torque 'T' is equal to 'tau max' multiplied by J divided by 'r'.

The maximum shear stress is equal to torque times the radius of the circular cross section divided by the polar moment of inertia.

Slide 14

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Polar Moment of Inertia, J

The **polar moment** of inertia **J** of a circle is similar to the moment of inertia **I_{xx}**, except in this case, as we are looking at the stresses acting over the whole area, we add **I_{xx}** and **I_{yy}** together:

$$J = I_{xx} + I_{yy}$$

The **polar moment of inertia** is used in all torsion formulas. It is twice the value of the Moment of Inertia of a circle

$$J = \frac{\pi d^4}{32} = \frac{\pi r^4}{2}$$

We have looked at the polar moment of inertia, J, in lecture 5a, slide number 14. J is the sum of the moment of inertia about the x axis and the moment of inertia about the y axis. We looked at how to derive the formula for calculating the moment of inertia in lecture 5a, slide number 15. For circular cross section, it is pi times r to the power of 4 divided by 2.

Slide 15

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The Torsion Formula

The shear stress **τ** in a circular bar subjected to torsion, **T** at a distance **r** from the centre is:

$$\tau = \frac{Tr}{J}$$

This equation is known as the **torsion formula** and shows that the shear stress **τ** is proportional to the applied torque **T**, the radius **r** and inversely proportional to the polar moment of inertia, **J**

τ is in MPa (N/mm²)
T is in Nmm
r is in mm
J is in mm⁴

$$\tau = \frac{Tr}{J} = \frac{\text{Nmm} \cdot \text{mm}}{\text{mm}^4} = \frac{\text{N}}{\text{mm}^2} = \text{MPa}$$

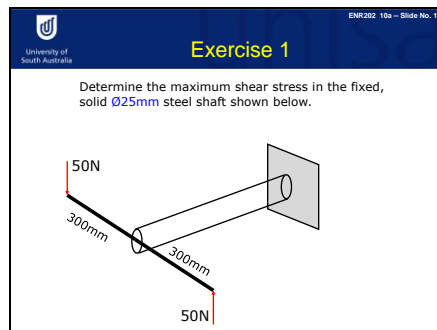
This equation is known as the torsion formula. It shows that shear stress 'tau' is proportional to the applied torque 'T', the radius 'r', and inversely proportional to the polar moment of inertia 'J'. We know the units of tau are stress units: that is, Mega Pascals or Newton per square mm. The torque unit is Newton mm, the radius is mm, the polar moment of inertia is mm to the power of 4. If you substitute the units into the formula, we get stress units for T multiplied by r divided by J (that is, Mega Pascals).

You can see that this formula is similar to the bending stress formula. Let's look at this in more details. We know the bending stress formula is moment M times y divided by the moment of inertia. The shear stress formula for torque is torque times r divided by the polar moment of inertia.

We calculate bending stress in bending and we calculate shear stress in torque. The moment of inertia in the bending formula is similar to the polar moment of inertia in the torsion formula. Y in

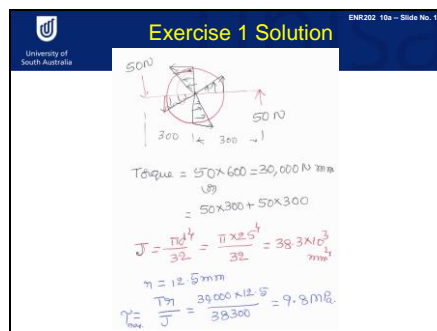
the bending formula is similar to the radius in the torque formula. The bending moment M in the bending formula is similar to the torque ' T ' in the torsion formula.

Slide 16



Suppose that a solid steel shaft has a diameter of 25 mm and is subjected to a couple force of 50 newtons at a distance of 300 mm from the centre of the cross section of the steel shaft, as shown in the figure. You need to calculate the maximum shear stress at the fixed end. Pause the presentation here and work this out. The solution is on the next slide.

Slide 17



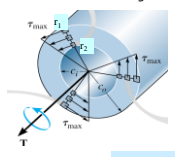
In this question, you are required to calculate the maximum shear stress on a solid steel shaft due to a torque. First, you need to calculate the torque. The torque is equal to force multiplied by distance, which is 50 newtons multiplied by 600 mm, which is equal to 30,000 Newton mm. Now calculate the polar moment of inertia. The diameter of the solid shaft is 25 mm, and the radius is equal to 12.5 mm. The polar moment of inertia is π times the diameter to the power of 4 divided by 32, which is equal to 38.3 multiplied by 10 to the power of 3 mm to the power of 4. The maximum shear stress occurs at the circumference, so the radius of

the shaft is 12.5 mm (that is, the distance from the centre of circle to the circumference). The maximum shear stress is torque multiplied by radius divided by the polar moment of inertia. If you substitute T, r and J into the torsion formula, you will get the value of 9.8 Mega Pascals or Newtons per square mm.

Slide 18

The Torsion Formula
Hollow Sections

Hollow circular bars: the analysis of the torsion of a hollow circular bar is almost identical to that for a solid bar, only the radial distance r appearing in these expressions is limited to the range of r_1 to r_2 and J equal to:


$$\tau_{\max} = \frac{Tr_2}{J}$$
$$\tau_i = \frac{Tr_1}{J}$$
$$\frac{\tau_{\max}}{\tau_i} = \frac{r_2}{r_1}$$
$$J = \frac{\pi}{2}(r_2^4 - r_1^4) = \frac{\pi}{32}(d_2^4 - d_1^4)$$

We have looked at the torsion formula for a solid circular cross section, but what about a hollow circular cross section? ‘co or r1’ is the radius of the outer circular cross section and ‘ci or r2’ is the inner circular cross section, as shown in the figure. The shear stress at the outer circle cross section is ‘tau max’ and the shear stress at the inner circle cross section is ‘tau i’. The shear stress distribution direction is the same as the applied torque direction.

The polar moment of inertia for a hollow circular shaft is the outer diameter to the power of 4 minus the inner diameter to the power of 4 times pi divided by 32, as shown in the equation. The maximum shear stress is equal to the torque times the outer radius divided by the polar moment of inertia. The shear stress at the inner circle is the torque times the inner radius divided by the polar moment of inertia. So, maximum shear stress divided by shear stress at the inner circle is equal to the outer radius divided by the inner radius. Therefore, the shear stress distribution is a linear variation.

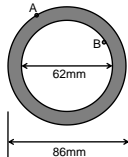
Slide 19

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Exercise 2

- Find the shear stress at points A and B on the shaft if it is subjected to a torque of 12kNm



The diagram shows a cross-section of a hollow circular shaft. The outer diameter is labeled as 86mm and the inner diameter is labeled as 62mm. Point A is located on the outer circumference, and point B is located on the inner circumference.

A hollow circular steel shaft has an outer diameter of 86 mm and an inner diameter 62 mm and is subjected to a torque of 12 kilo Newtons meters, as shown in the figure.

You are required to calculate the shear stress at point A and point B, as shown in the figure. Point A is located on the outer circle and point B is located on the inner circle. Pause the presentation and work out this problem now. The solution is on the next slide.

Slide 20

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Exercise 2 Solution

- Calculate J: $J = \frac{\pi d^4}{32} = \frac{\pi (86^4 - 62^4)}{32} = 3.92 \times 10^6 \text{ mm}^4$
- Calculate Maximum Shear Stress at Points A and B:
 - $T = 12 \text{ kNm} = 12 \times 10^6 \text{ Nmm}$
 - At A, $r = 43 \text{ mm}$; At B, $r = 31 \text{ mm}$
 - Shear stress A: $= \frac{12 \times 10^6 \times 43}{3.92 \times 10^6} = 131.6 \text{ MPa}$
 - Shear stress B: $= \frac{12 \times 10^6 \times 31}{3.92 \times 10^6} = 94.9 \text{ MPa}$

First, calculate the polar moment of inertia of the hollow circular shaft. The outer diameter is 86 mm and the inner diameter is 62 mm. The polar moment of inertia 'J' is equal to $\frac{\pi}{32} (86^4 - 62^4)$, which is 3.92 times 10 to the power of 6 mm to the power of 4. The torque here is already given – it is 12 kilo Newton meters. Convert this value into Newton mm, because the J value is in mm to the power of 4.

The shear stress on the outer circle at point A is equal to the torque multiplied by the outer circle radius, divided by the polar moment of inertia, which is 12.6 times 10 to the power of 6 multiplied by the outer circle radius (43 mm) divided by the polar moment of inertia which is 3.92 times 10 to the power of 6. Thus, we get the shear stress on the outer circle at point A as being equal to 131.6 Mega Pascals.

The shear stress on the inner circle at point B is equal to the torque multiplied by the inner circle radius divided by the polar moment of inertia, which is 12.6 times 10 to the power of 6 multiplied by the inner circle radius (31 mm) divided by the polar moment of inertia, which is 3.92 times 10 to the power of 6. Thus, we get the shear stress on the inner circle at point B as being equal to 94.9 Mega Pascals or Newtons per square mm.

Slide 21

ENR202 10a - Slide No. 21

Angle of Twist Formula

If the shaft's cross-sectional area and the applied torque are constant along the length of the shaft, the total angle of twist ϕ is equal to:

$$\gamma L = BB' \quad \gamma = \frac{\tau}{G} \quad \text{By Hooke's law} \quad \phi = \frac{\tau L}{Gr} \quad \tau = \frac{Tr}{J}$$

$$r\phi = BB' \rightarrow \gamma L = r\phi \rightarrow \phi = \frac{TL}{GJ}$$

Where:

- ϕ = Angle of twist (Radians)
- T = torque (Nmm) ***
- L = Length of shaft (mm)
- J = Polar moment of Inertia (mm⁴)
- G = Shear Modulus (MPa)

How do we calculate the angle of twist (FI) for a circular shaft? We have already looked at some basics on the angle of twist, in slide 9 of this lecture, where we looked at the very small segment length of dx.

Now we will consider the whole length of the shaft, and calculate the distance between b and b dash, which is equal to r times 'FI', where r is the radius. We can also calculate the distance between b and b dash with respect to the length, which is 'gamma' times L. Therefore, based on these two calculations, we work out that 'gamma' times 'L' is

equal to 'r' times 'FI'. Using Hooke's law, shear strain is equal to shear stress divided by shear modulus. If you substitute shear strain γ into the equation, we get the angle of twist as being equal to τ times L divided by the shear modulus times the radius. The torsion formula states that the shear stress is equal to the torque times the radius divided by J. Substitute the torsion formula into the previous angle of twist formula. You will get the angle of twist being equal to T times L divided by GJ.

Slide 22

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Exercise 3

The steel pipe has an inner diameter of 80mm and an outer diameter of 100mm. If its end is tightened against the support at A using a torque wrench at B, determine the shear stress developed in the material at the inner and outer walls of the pipe when the 80N couple is applied to the wrench. Find the angle of twist if the shaft is 800mm long. $G = 80,000\text{MPa}$

Figure 1

Figure 2

Figure 3

We will now look at an exercise based on the angle of twist formula. This steel pipe has an inner diameter of 80 mm and an outer diameter of 100 mm. The support A is fixed, as shown in figure 1. You need to determine the shear stress in the material at the inner and outer walls of the pipe when the 80 Newton couple is applied at a distance of 500 mm, as shown in the figure 1. You also need to calculate the angle of twist if the shaft is 800 mm long (that means the distance between points A and B). The shear modulus is 80 thousands Mega Pascals. Pause the presentation to work this out. The solution is on the next slide.

Slide 23

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ENR202 10a - Slide No. 23

Exercise 3 Solution

Step 1- Calculate $J = \pi * (100^4 - 80^4) / 32 = 5.79 * 10^6 \text{ mm}^4$

Step 2- Torque $T = 80 \text{ N} * 200 \text{ mm} + 80 \text{ N} * 300 \text{ mm} = 40,000 \text{ Nmm}$

Step 3- Calculate Shear Stress at outer and inner faces

$r_1 = 50 \text{ mm}, r_2 = 40 \text{ mm}$

Shear outer $= 40,000 * 50 / 5.8 * 10^6 = 0.345 \text{ MPa}$

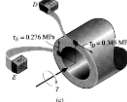
Shear inner $= 40,000 * 40 / 5.8 * 10^6 = 0.276 \text{ MPa}$

Step 4- Calculate angle of twist

$= 40000 * 800 / 5.8 * 10^6 \text{ mm}^4 * 80000 = 6.9 * 10^{-5} \text{ Radians}$
(or $6.9 * 10^{-5} \text{ Rads} * (180/\pi) = 0.004 \text{ degrees}$)

$\tau = \frac{T\rho}{J}$

$\theta = \frac{TL}{JG}$



First, calculate the polar moment of inertia of the hollow shaft. The outer diameter is 100 mm and the inner diameter is 80 mm. J is 100 mm to the power of 4 minus 80 mm to the power of 4 times pi divided by 32, which is 5.79 times 10 to the power of 6 mm to the power of 4.

The second step is to calculate the torque T , which is equal to the force 80 Newtons multiplied by a distance of 200 mm plus the other equal force of 80 Newtons multiplied by 300 mm, which is equal to 40 thousands Newton mm.

The third step is to calculate the shear stress at the outer wall of the pipe, which is equal to the torque multiplied by the outer wall radius divided by the polar moment of inertia, which is equal to 0.345 Mega Pascals, and the inner wall of the pipe, which is equal to the torque multiplied by the inner wall radius divided by the polar moment of inertia, which is equal to 0.276 Mega Pascals. You can see the shear stress distribution of these values in the figure.

For the fourth step, calculate the angle of twist of the pipe, which is equal to the torque of 40 000 Newton mm times the length of the pipe, which is 800 mm divided by the shear modulus of 80 000 Mega Pascals times the polar moment of inertia, which is 5.8 times 10 to the power of 6 mm to the power of 4, which is equal to 6.9 times 10 to the power of minus 5 radians. Convert radians into degrees, and the angle of twist is 0.004 degrees.

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Multiple Angles of Twist

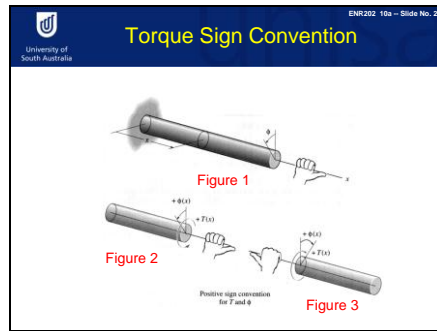
If the shaft is subjected to several different torques, or the cross-sectional area or shear modulus changes abruptly from one region to the next, then the total amount that the shaft rotates can be calculated using the angle of twist formula for each part and the summing the results:

$$\phi_{Total} = \sum \frac{TL}{JG}$$

The diagram shows a horizontal shaft with four segments: GC, CA, AE, and EF. Segment GC has a length of 1.2 m and a torque of 60 kW acting clockwise at point C. Segment CA has a length of 2 m and a torque of 20 kW acting anti-clockwise at point D. Segment AE has a length of 1.2 m and a torque of 30 kW acting clockwise at point E. Segment EF has a length of 2 m and a torque of 70 kW acting anti-clockwise at point F. The shaft is supported by bearings at points B and G.

Now we will look at how to calculate the angle of twist if the shaft is subjected to several different torques, or if there are different cross sectional areas or different materials in the same shaft. We have several torques acting as shown in the figure. We have supports at Point B and G. At point C, a torque of 60 kilowatts is acting in a clockwise direction. At point D, there is a 20 kilowatt torque acting in an anti-clock wise direction. At point E, there is a 30 kilowatt torque acting in a clockwise direction. At point F, there is a 70 kilo watt torque acting in anti clockwise direction. In this type of case, we need to calculate the sum of all twists in each region: GC, CA, AE and EF. The only condition that you need to consider is the division of the region in such a way that the torque T, the polar moment of inertia J, and the shear modulus G have the same values in that region.

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Before you work on a multiple angle of twist problem, you need to follow the sign conventions for torque. Use the Direction of Right Hand Thumb rule. Say a circular shaft is subjected to torque at one end and fixed at other end. You cut the shaft at a distance 'x' from the fixed end, as shown in figure 1. Consider the left shaft of the cutting cross section. Hold your right hand thumb away from the cutting cross section as shown in figure 2 and the positive torque direction is towards the other fingers. If you apply torque in that direction, that torque is positive. Otherwise, it is negative.

Now consider the right shaft of the cutting cross section. Hold your right hand thumb away from the cutting cross section, as shown in figure 3. The positive torque direction is towards the direction of the other fingers. If you apply torque in that direction, that torque is positive. Otherwise, it is negative.

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Exercise 4

The gears attached to the fixed-end (at A) steel shaft are subjected to the torques shown in the figure. If the shear modulus of elasticity is $G=80000\text{MPa}$ and the shaft has a diameter of 14mm, determine the displacement of the tooth P on gear A (radius $A=100\text{mm}$). The shaft turns freely within the bearing at B.

(a)

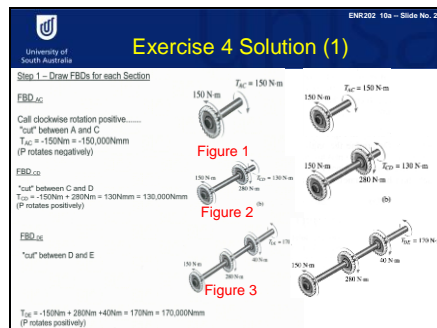
Let's try another exercise. A steel shaft is supported at point B and fixed at point E, as shown in the figure. The shaft is subjected to the following torques:

- an anti-clockwise torque of 150 Newton meters at the gear at point A, which has 100 mm radius
- a clockwise torque of 280 Newton meters at point C
- a clockwise torque of 40 Newton meters at point D

The shear modulus of the shaft is 80 thousand Mega Pascals, and its diameter is 14 mm.

Pause this presentation and work out the displacement of tooth P on gear A. (Note that the shaft turns freely within the bearing at B.) The solution is on the next three slides.

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First, draw the Free Body Diagram for each section. We have 3 sections: AC, CD and DE. In the AC section, you cut the shaft between point A and point C, and draw the free body diagram as shown in figure 1. Apply the equilibrium equation and you will get the torque in section AC as being equal to 150 Newton meters. Due to T_{AC} P, the point at gear A rotates negatively, so this torque sign is negative.

Next, cut shaft between point C and D and draw the free body diagram as shown in figure 2. Apply the equilibrium equation to get the torque in section CD as equal to 130 Newton meters. Due to T_{CD} , point P at gear A rotates positively, so this torque sign is positive.

Finally, cut shaft between points D and E and draw the free body diagram as shown in figure 3. Apply the equilibrium equation and you will get the torque in section DE as equal to 170 Newton meters. Due to T_{DE} , point P at gear A rotates positively, so this torque sign is positive.

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Exercise 4 Solution (2)

Step 2 – Determine Polar Moment of Inertia J
 $J = \frac{\pi D^4}{32} = \frac{\pi \cdot 14^4}{32} = 3.77 \cdot 10^3 \text{ mm}^4$

Step 3 – Calculate and Sum Individual segment angles $\phi = \frac{TL}{JG}$

Segment A-C
 $= -150000 \cdot 400 / 3.77 \cdot 10^3 \cdot 80000 = -0.198 \text{ Radians (anti CW)}$

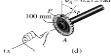
Segment C-D
 $= +130000 \cdot 300 / 3.77 \cdot 10^3 \cdot 80000 = 0.129 \text{ Radians (CW)}$

Segment D-E
 $= +170000 \cdot 500 / 3.77 \cdot 10^3 \cdot 80000 = 0.282 \text{ Radians (CW)}$

$\phi_{\text{Total}} = \sum \frac{TL}{JG} = -0.198 + 0.129 + 0.282 = 0.213 \text{ Radians (CW)}$

Step 4 – Calculate Displacement

Displacement $= \theta r = 0.213 \cdot 100 = 21.2 \text{ mm}$



For the second step, you need to calculate the polar moment of inertia for the shaft. The shaft has same diameter throughout the length: 14 mm. The polar moment of inertia for this solid shaft is pi times 14 mm to the power of 4 divided by 32, which is 3.77 times 10 to the power of 3 mm to the power of 4.

For the third step, you need to calculate and find the sum of the individual section angles of twist. This will give you the angle of twist at point P at gear A with respect to point E. We have looked at how to calculate the angle of twist if torque T, polar moment of inertia J, and shear modulus G have the same value on a particular length of the shaft. The AC section is 400 mm long and the torque is negative 150 000 newton mm. The J value is the same for all sections and has already been calculated in step 2. The shear modulus is also the same for the whole pipe and is given in the problem (80 000 Mega Pascals). The angle of twist for the AC section is minus 0.198 radians in an anti-clockwise direction.

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Exercise 4 Solution (3)

Step 2 – Determine Polar Moment of Inertia J
 $J = \frac{\pi D^4}{32} = \frac{\pi \cdot 14^4}{32} = 3.77 \cdot 10^5 \text{ mm}^4$

Step 3 – Calculate and Sum Individual segment angles $\phi = \frac{TL}{JG}$

Segment A-C
 $= -150000 \cdot 400 / 3.77 \cdot 10^5 \cdot 80000 = -0.198 \text{ Radians (anti CW)}$

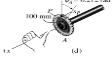
Segment C-D
 $= +130000 \cdot 300 / 3.77 \cdot 10^5 \cdot 80000 = 0.129 \text{ Radians (CW)}$

Segment D-E
 $= +170000 \cdot 500 / 3.77 \cdot 10^5 \cdot 80000 = 0.282 \text{ Radians (CW)}$

$\phi_{\text{Total}} = \sum \frac{TL}{JG} = -0.198 + 0.129 + 0.282 = 0.213 \text{ Radians (CW)}$

Step 4 – Calculate Displacement

Displacement = $\theta r = 0.213 \cdot 100 = 21.2 \text{ mm}$



The CD section is 300 mm long and a positive torque of 130 000 Newton mm. The J value is the same for all sections and has already been calculated in step 2. The shear modulus is also the same for the whole pipe (80 000 Mega Pascals). The angle of twist for the CD section is positive 0.129 radans in a clockwise direction.

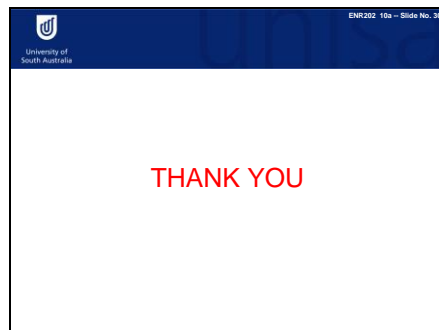
The DE section is 500 mm long and has a positive torque of 170 000 Newton mm. The J value is the same for all sections and has already been calculated in step 2. The shear modulus is also the same for the whole pipe (80 000 Mega Pascals). The angle of twist for the DE section is positive 0.282 radians in a clockwise direction.

The sum of the angle of twist from all sections is minus 0.198 plus 0.129 plus 0.282, which is positive 0.213 radians in a clockwise direction.

Finally, you work out that the displacement of point P at gear A in a clockwise direction is 0.213 times the radius of gear A which is

100 mm. The answer is 21.2 mm, as shown in the figure.

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In the next lecture summary, we will continue looking at the concept of torsion, but focusing on power transmission and thin-walled tube cross sections. Thank you for your attention.