Logarithms

Section 1: Definition

We come now to the question of what a logarithm is. We must remind ourselves first of the power notation and its effects. Remember that when n is an integer, $a^n = a.a....a$, with n factors of a.

Example 1

(a) $2^4 = 2.2.2.2 = 16$

(b) $10^5 = 10.10.10.10.10 = 100{,}000$

(c) $2^6 = 2.2.2.2.2.2 = 64$

(d) $10^3 = 10.10.10 = 1{,}000$

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We need to remember that if n is a negative power, then we take the reciprocal, ie $a^{-n} = \dfrac{1}{a^n}$. (Assuming n to be positive, although it works just as well if n is negative.) Also note that if either a or n is not an integer, then we would evaluate a^n using a calculator, Excel, Matlab or some other software. Check the following using your calculator.

Example 2

(a) $2^{1.5} = 2.828$

(b) $10^{3.47} = 2{,}951.2$

(c) $0.942^{3.8} = 0.797$

(d) $4.6^{-1.6} = 0.0870$

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We are now able to define the term *logarithm*. What is the logarithm of x? In fact we cannot answer this question unless another number a, known as the base, is also specified. Thus the question, what is the logarithm of x is meaningless, unless we make an assumption about the base a, whereas the question what is the logarithm of x base a is very meaningful. We shall define the logarithm by defining it in terms of its *inverse*.

Suppose that a number $x = a^y$, where a is the base specified and y is either known or can be found out somehow. Then we say that the logarithm base a of x is y, and write $y = \log_a(x)$.

Example 3

(a) $32 = 2^5; \log_2(32) = 5$

(b) $16 = 2^4; \log_2(16) = 4$

(c) $2^0 = 1; \log_2(1) = 0$

(d) $2^{-3} = 0.125; \log_2(0.125) = -3$

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Now complete the following table of logarithms base 2. Remember that a logarithm is actually a power. The answers are at the end of this section.

Table 1

Exponential form	*Logarithmic form*
$2^5 = 32$	$\log_2 32 = 5$
$2^4 = 16$	$\log_2 16 =$
$2^3 = 8$	$\log_2 8 =$
$2^2 = 4$	$\log_2 4 =$
$2^1 = 2$	$\log_2 2 =$
$2^0 = 1$	$\log_2 1 =$
$2^{-1} = 0.5$	$\log_2 0.5 =$
$2^{-2} = 0.25$	$\log_2 0.25 =$

The same thing can be done with powers of 10. Complete the following table and find the logarithms (base 10) of the numbers given.

Table 2

Exponential form	*Logarithmic form*
$10^5 = 100{,}000$	$\log_{10} 100{,}000 = 5$
$10^4 = 10{,}000$	$\log_{10} 10{,}000 =$
$10^3 = 1{,}000$	$\log_{10} 1{,}000 =$
$10^2 = 100$	$\log_{10} 100 =$
$10^1 = 10$	$\log_{10} 10 =$
$10^0 = 1$	$\log_{10} 1 =$
$10^{-1} = 0.1$	$\log_{10} 0.1 =$
$10^{-2} = 0.01$	$\log_{10} 0.01 =$

We can do the same with other numbers that are not such convenient powers, ie non-integer powers of the base. Try the following table. Once again, the answers are at the end of this section.

Table 3

Exponential form	*Logarithmic form*
$10^{4.87} = 74{,}131.0$	$\log_{10} 74{,}131.0 =$
$10^{2.8} = 630.96$	$\log_{10} 630.96 =$
$10^{0.54} = 3.467$	$\log_{10} 3.467 =$
$10^{-0.98} = 0.105$	$\log_{10} 0.105 =$
$10^{-2.73} = 0.00186$	$\log_{10} 0.00186 =$

In fact there are only two really common bases[1], base 10 and base *e*, where *e* is a special number like π. We shall consider this base later. In fact logarithms base 10 are so common, that we shall assume that if the base is not specified via a subscript then it is automatically base 10. Hence from now on, $\log(x) = \log_{10}(x)$. Logarithms base *e* are even more common and have an even shorter notation that we shall see soon.

Answers
Table 1: 4, 3, 2, 1, 0, -1, -2
Table 2: 4, 3, 2, 1, 0, -1, -2
Table 3: 4.87, 2.8, 0.54, -0.98, -2.73

Section 2: Properties of Logarithms

The properties of logarithms, or logs, logs are similar to, and derive from, the properties of powers. Hence for every property of powers, or indices, there is a corresponding property of logs. These properties will not be justified, but they flow fairly easily from the corresponding property of powers, and the reader is referred to Calter and Calter[2], chapter 20, for the justification. The properties are formulated in terms of some arbitrary base *b.*

Table 4

Property of Powers	*Property of Logs*
$b^x b^y = b^{x+y}$	$\log_b MN = \log_b M + \log_b N$
$\dfrac{b^c}{b^d} = b^{c-d}$	$\log_b\left(\dfrac{M}{N}\right) = \log_b M - \log_b N$
$\left(b^c\right)^d = b^{cd}$	$\log_b M^p = p\log_b M$

[1] The exception to this rule is in computer science, where powers of two are so important that logarithms base 2 are frequently used, and given a special notation: $\log_2(x) = \lg(x)$.
[2] Technical Mathematics With Calculus, Calter and Calter.

$\sqrt[p]{M} = M^{\frac{1}{p}}$	$\log_b \sqrt[p]{M} = \log_b (M)^{\frac{1}{p}} = \frac{1}{p}\log_b M$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
	$\log_b (b^p) = p\log(b) = p$
$y = b^x; x = \log_b y$ (by definition)	$b^{\log_b x} = x$
(change of base)	$\log_b N = \frac{\log N}{\log b} = \frac{\ln N}{\ln b}$

There are plenty of examples of these properties in the reference, Calter and Calter.

Side note: What on Earth is e?

Most students understand relatively easily why 10 is/was such a common base for logarithms. The logs give the order of magnitude straight away. However logs base e do not have this advantage. So why use it? The answer to that is that so many problems involving powers or logs are problems of growth or decay, where some quantity is either growing or decaying. In problems such as finding the interest on a bank account, the times at which interest is paid are well spaced out, usually months and occasionally yearly. However many problems in nature require continuous growth or continuous decay. For example a heated object placed in cooler surroundings cools down continuously, rather than at discrete intervals. Hence mathematicians became interested in the constant that arose when the "interest periods" were allowed to shrink towards 0. The equation for compound interest (see Calter and Calter, section 20.2) became $y = a\left[\left(1+\frac{1}{k}\right)^k\right]^{nt}$. They then discovered that as k grew towards infinity, the expression $\left(1+\frac{1}{k}\right)^k$ actually grew to a fixed number, namely the number 2.71828182…, which they called e. Because these problems of continuous growth were so common, e became the default base for logarithms.

Section 3: Calculators, Logs, Antilogs and Equations

Modern calculators have greatly simplified the process of solving equations with powers or logs in them. It is literally just the touch of a button to find out a power or a log. The relevant buttons are usually marked log and ln for the logarithms base 10 and base e. The reverse operations, or antilogs, are usually marked as 10^x and e^x. Try the following examples on your own calculator. Solutions are below the table.

Table 5

Logs	*Powers*
$\log 47 = 1.672$	$10^{3.8} = 6{,}309.6$
$\log 4.7 =$	$10^{1.23} =$
$\log 0.276 =$	$10^{0.39} =$
$\log 1.375 =$	$10^{-0.23} =$
$\log 5{,}487 =$	$10^{-0.58} =$
$\ln 47 = 3.850$	$e^{3.8} = 44.70$
$\ln 1.35 =$	$e^{1.58} =$
$\ln 0.135 =$	$e^{0.27} =$
$\ln 2.35 =$	$e^{-0.64} =$

Solutions: log(4.7)=0.672, log(0.276)=-0.559, log(1.375)=0.138, log(5487)=3.739, ln(1.35)=0.300, ln(0.135)=-2.002, ln(2.35)=0.854.$10^{1.23}$=16.98, $10^{0.39}$=2.455, $10^{-0.23}$=0.5888, $10^{-0.58}$=0.2630, $e^{3.8}$=44.70, $e^{1.58}$=4.855, $e^{0.27}$=1.310, $e^{-0.64}$=0.5273.
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Both of these operations are essential to solving many problems that we will encounter. However it is essential to realise the operations of taking a logarithm and raising the base b to a power are inverse operations. We shall thus use them to solve equations in a similar way to solving linear equations When solving a linear equation, we apply inverse operations. For example to reverse a multiplication, we divide. To reverse an addition, we subtract. Consider the following examples.

Example 4

Solve the equation $10^{2x-3} = 5$. Now the inverse operation to taking a power to base 10 is taking the log. Hence we have

$$\log\left(10^{2x-3}\right) = 2x - 3 = \log(5) = 0.6990$$

$$2x = 3.6990$$

$$x = 1.850$$

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Example 5

$$10^{\frac{4x-3}{2}} = 14.8$$

$$\log\left(10^{\frac{4x-3}{2}}\right) = \frac{4x-3}{2} = \log(14.8) = 1.1703$$

$$4x - 3 = 2.3405$$

$$4x = 5.3405$$

$$x = 1.335$$

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Example 6

Solve the equation $e^{2x+1} = 11.9$. Now the reverse operation to taking a power, base *e*, is taking a natural logarithm, ie log base *e* or ln.

$$e^{2x+1} = 11.9$$

$$\ln\left(e^{2x+1}\right) = 2x + 1 = \ln(11.9) = 2.4765$$

$$2x + 1 = 1.4765$$

$$2x = 1.4765$$

$$x = 0.738$$

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Example 7

$$e^{\frac{3}{x+2}} = 15$$

$$\ln\left(e^{\frac{3}{x+2}}\right) = \frac{3}{x+2} = \ln(15)$$

$$x + 2 = \frac{3}{\ln(15)}$$

$$x = \frac{3}{\ln(15)} - 2 = -0.892$$

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Of course we can also solve equations involving logarithms. The basis for this is as follows: if A=B then

$$10^A = 10^B ; 10^{\log(A)} = A;$$

$$e^A = e^B ; e^{\ln(A)} = A$$

Example 8

$$\log(2x + 5) = 0.8$$

$$10^{\log(2x+5)} = 2x + 5 = 10^{0.8} = 6.310$$

$$2x = 1.310$$

$$x = 0.655$$

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Example 9

$$\ln\left(\frac{2}{x-3}\right)=2.1$$

$$e^{\ln\left(\frac{2}{x-3}\right)}=\frac{2}{x-3}=e^{2.1}=8.166$$

$$\frac{x-3}{2}=\frac{1}{8.166}=0.1225$$

$$x-3=0.2449$$

$$x=3.245$$

♦ ♦ ♦[3]

Example 10

$$\ln\left(\frac{5x-1}{7}\right)=3$$

$$e^{\ln\left(\frac{5x-1}{7}\right)}=\frac{5x-1}{7}=e^{3}$$

$$5x-1=7e^{3}$$

$$5x=1+7e^{3}$$

$$x=\frac{1+7e^{3}}{5}=28.32$$

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Try following the above examples through to the end without writing down any intermediate results, as shown in Example 10.

Section 4: Applications

Two common applications of this sort of theory are growth and decay problems. In growth problems, some quantity, such as a population of bacteria, or the volume of infected emails, is growing exponentially. That is to say, they will follow a sort of relationship of the form $A=A_0e^{kt}$, where A is the current (or future) amount, A_0 is the initial amount, k is a (positive) constant and t is the time (since the beginning of the experiment, for want of a better word). Typically one will be given the amount A_0 at the start, when t=0, and at some later time and be asked to find the time required for A to double. Or one may be asked to find the time when the amount will reach a certain level. These are relatively simple examples of the equations solved above. Consider the following example.

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[3] Note that really the above examples are a trifle misleading. Numbers such as $e^{2.1}$ should not be evaluated until the last possible moment, so that the solution to the above example becomes $x=2/(e^{2.1})+3$. This will lead to greater accuracy.

Example 11

A virus is released upon an unsuspecting world by the sending of 10 infected emails. Five hours later, there are estimated to be 2,000 infected emails. Find how long it is taking for the number of emails to double, and predict when the number of infected emails will reach 20,000.

$A = A_0 e^{kt}$ and $A_0 = 10$, and at $t = 5$, A=2000

$2000 = 10e^{5k}$

$e^{5k} = 200$

$5k = \ln(200)$

$k = \dfrac{\ln(200)}{5}$ $(=1.060)$

So $A = 10e^{1.060t}$

Now the time taken to double will be found by answering the question: When is $A=2A_0$?

$2A_0 = A_0 e^{1.060t}$

$2 = e^{1.060t}$

$1.060t = \ln(2)$

$t = \dfrac{\ln(2)}{1.060} = 0.654$

ie the number of infected emails is doubling every 0.654 hours, or every 39 minutes, 14 seconds.

To answer the question: when will the number of infected emails reach 20,000, we must solve another equation, this time with t unknown.

$20000 = 10e^{1.060t}$

$2000 = e^{1.060t}$

$1.060t = \ln(2000)$

$t = \dfrac{\ln(2000)}{1.060} = 7.171$

ie there should be 20,000 infected emails after 7.171 hours, or 7 hours and 10 minutes.

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Decay problems are very similar, but revolve around quantities that are decreasing, typically the temperature of a cooling object or the amount of some radioactive substance. The equation to be used for these problems is almost the same: $A = A_0 e^{-kt}$, where k is again assumed to be positive.

References

Calter and Calter, Technical Mathematics With Calculus, 4th edition. (Chapter 20).