Theorem 5.2.14

Theorem

Consider the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$. We know that the characteristic equation of this is $t^2 - c_1 t - c_2$. If r is a repeated root to this characteristic equation, then apart from the solution r^n, we also have the solution $a_n = nr^n$.

Proof

Trying the solution $a_n = nr^n$, we also find that $a_{n-1} = (n-1)r^{n-1}$, $a_{n-2} = (n-2)r^{n-2}$.
So the right hand side of the recurrence relation becomes

$c_1 a_{n-1} + c_2 a_{n-2} = c_1(n-1)r^{n-1} + c_2(n-2)r^{n-2}$

$= c_1 nr^{n-1} - c_1 r^{n-1} + c_2 nr^{n-2} - 2c_2 r^{n-2}$

$= n(c_1 r^{n-1} + c_2 r^{n-2}) - (c_1 r^{n-1} + 2c_2 r^{n-2})$

$= nr^n - (c_1 r^{n-1} + 2c_2 r^{n-2})$

(as r^n is a solution to the recurrence relation)

Now we have run into a bit of a snag, and need to find some more information about c_1 and c_2. We remember that the characteristic equation was

$t^2 - c_1 t - c_2 = 0.$

On the other hand, we also know that r was a repeated root of that equation.
So we can confidently say that the equation must also be

$(t-r)^2 = 0$ or

$t^2 - 2rt + r^2 = 0.$

Hence, by comparing the t term and the constant term in our two equations,

$2r = c_1$ and $-c_2 = r^2$

So $c_1 r + 2c_2 = 2r^2 - 2r^2 = 0$. Hence

$nr^n - (c_1 r^{n-1} + 2c_2 r^{n-2}) = nr^n - 0 = nr^n$.

But this is exactly what we were trying to prove, as it is what the right hand side is. So we accept that $a_n = nr^n$ is a valid solution to the original recurrence relation.

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