Related Rates

This is one of the more difficult concepts in Essential Mathematics 2. I think that is because it is so different and unusual, compared to what we have been doing up until now. It takes a relationship between two quantities, perhaps *x* and *y*, and introduces a third variable, time *t*, which does not appear in the equation at all. We then find the relationship between the rates of change of our original two variables. It requires us to perform implicit differentiation, not with respect to one of the variables present in the equation, but one that is not (explicitly) present. Let us illustrate with a few examples.

Example 1

(a) Suppose $y^2 = 2x+3$. Then differentiating implicitly with respect to *x*, we would find $2y\frac{dy}{dx} = 2, \frac{dy}{dx} = \frac{1}{y}$. We have not introduced a third variable, *t*. We are still, at this point, assuming *x* to be the independent variable and *y* the dependent variable.

(b) Suppose $\sin(y)+2 = \cos(x)+x^2$. Differentiating with respect to x, we find $\cos(y)\frac{dy}{dx} = -\sin(x)+2x, \frac{dy}{dx} = \frac{2-\sin(x)}{\cos(x)} = 2\sec(x)-\tan(x)$.

(c) Suppose $x^2+2xy+y^3 = 4$. Differentiating with respect to *x*, we find $2x+2y+2x\frac{dy}{dx}+3y^2\frac{dy}{dx} = 0, \frac{dy}{dx} = \frac{-2(x+y)}{2x+3y^2}$.

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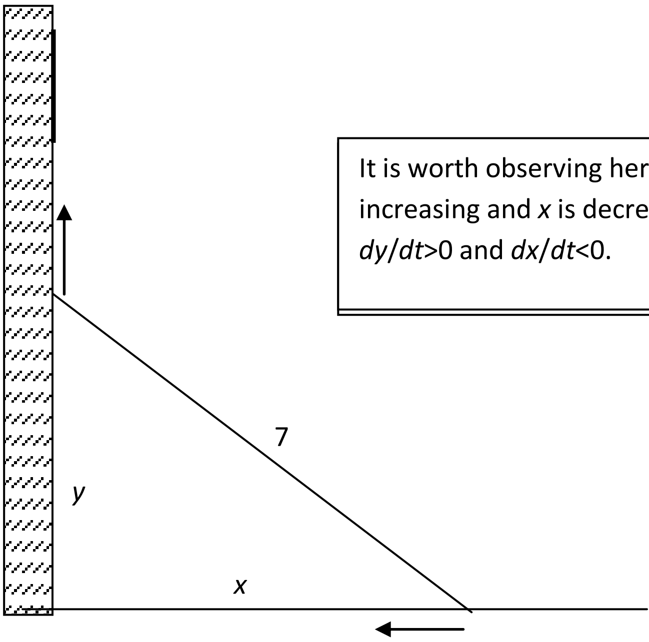
Now we are going to introduce the variable time. That is to say, even though time will NOT appear explicitly in our equation, we understand that BOTH quantities are changing over time and hence are dependent on time. So *t* is the independent variable, even though it does not appear in the equation, and the other two variables are dependent on *t*.

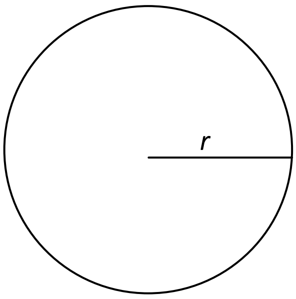
Example 2

(a) A ladder of length 7 metres is being used to access a window on a building site. The ladder needs to be raised to the height of the window, so a rope is passed out of the window, lowered down and tied around the top of the ladder. The rope is then hauled up until the top of the ladder is at the correct height. If *x* is the distance from the foot of the building, and *y* is the height of the top of the ladder, then the diagram below shows this situation.



It is worth observing here that *y* is increasing and *x* is decreasing, that is $dy/dt>0$ and $dx/dt<0$.

(b) An oil spill occurs at sea and the oil immediately begins to spread out in a circular fashion. The area of the spill is given by $A = \pi r^2$, and both area and radius are increasing over time. Hence *A* and *r* are both implicitly dependent on time.



Here both *r* and *A* are increasing, that is $dA/dt>0$ and $dr/dt>0$.

(c) A sphere of bath salts is dropped into a hot bath. The sphere dissolves in the hot water, meaning that both its volume and radius are decreasing. Now $V = \frac{4}{3}\pi r^3$ and *V* and *r* are both functions of time. As the volume of the sphere is lost to the bath water, the radius of the sphere must decrease. Hence in this case both $\frac{dV}{dt}$ and $\frac{dr}{dt}$ must be negative.

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Now we shall try a few examples. First we shall just differentiate with respect to the independent variable *t*. After that we shall try solving some actual problems with numbers.

Example 3

(a) $x^2+y^2=L^2$, where *L* is a constant. $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$. This now has 4 unknown quantities in it, so if we are given values for 3 of them we can find the fourth.

(b) $A=\pi r^2, \frac{dA}{dt}=2\pi r\frac{dr}{dt}$, and this has 3 quantities in it. If we are given values for 2 of them we can solve for the third.

(c) $V=x^3, \frac{dV}{dt}=3x^2\frac{dx}{dt}$ and this also has three quantities in it. If we are given values for 2 of them we can solve for the third.

(d) $V=\frac{4}{3}\pi r^3, \frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$ and this also has three quantities.

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Example 4

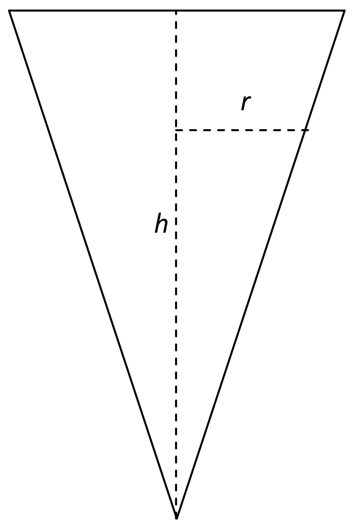
(a) An oil spill is increasing in radius at the rate of 600 metres per hour. Find the rate of increase of the area when the radius is 2 km.

Note that, as above, $A=\pi r^2, \frac{dA}{dt}=2\pi r\frac{dr}{dt}$. We know that $\frac{dr}{dt}=600$ metres per hour, so knowing that *r*=2000 we find $\frac{dA}{dt}=2\pi r\frac{dr}{dt}=2\pi*2000*600=2{,}400{,}000\pi\approx 7{,}539{,}822.4$ square metres per hour. What this enables us to say is that if the spill continues to increase *at this rate* then in one more hour it will have increased in size by just over 7.5 million square metres.

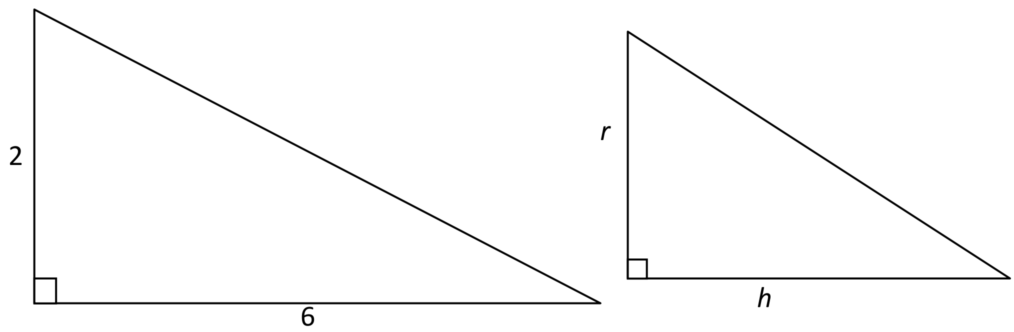
(b) An ice cube is melting in such a way that all sides are decreasing at the same rate. (Technically this unlikely, since the bottom side would be on some sort of plate, so it is either warmer or cooler than the other sides, but for the sake of the problem we will make this assumption anyway.) The sides are decreasing at the rate of 2 cm per hour. Find the rate of decrease of the volume when the sides are 20 cm long.

$V = x^3$, $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$. Note that $\frac{dx}{dt} = -2$, so $\frac{dV}{dt} = 3*20^2*(-2) = -2400$ cubic centimetres per hour.

(c) A conical tank of radius 2 metres and height 6 is draining of water. The water is leaving the tank at a rate of 0.8 cubic metres per second. Find the rate of change of the height of the water when the height is 3 metres.



This gives us two right triangles, as follows.



These are similar triangles, in that the sides are parallel to each other, even though the sides are of different lengths. Because they are similar, we have the following.

$$\frac{r}{2} = \frac{h}{6}$$

The volume of the tank is given by $V = \frac{1}{3}\pi r^2 h$, which contains three variables not 2. We need to eliminate one of those, say *h*.

$$r = \frac{2h}{6} = \frac{h}{3}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 * h = \frac{\pi}{27}h^3$$

Now we can differentiate both sides with respect to time, as before.

$$\frac{dV}{dt} = \frac{\pi}{9}h^2 \frac{dh}{dt}$$

Of course $\frac{dV}{dt} = -0.8$, and $h = 3$.

$$-0.8 = \frac{\pi}{9}3^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-0.8}{\pi} \approx -0.2546$$ metres per hour.

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